Graphs and Networks **out:** October 5, 2007

Problem Set 2

Lecturer: Daniel A. Spielman **due:** October 18, 2007

1 Introduction

There are 3 sections to this problem set. The first contains the 5 problems for the undergraduates. All have short answers. The second contains an additional problem for the graduate students. The third contains the optional experimental portion. The experimental portion may be substituted for **3** problems.

2 Homework Policy

You may discuss the problems with other students. But, you must write your solutions independently, drawing on your own understanding. You should cite any sources that you use on the problem sets other than the textbook, TA and instructor. This means that you should list your collaborators.

You **may not** search the web for solutions to similar problems given out in other classes. If you think this policy needs any clarification, please let me know.

3 Corrections Made

I've reversed the inequality in problem 2: it should be $\lambda_i \leq \gamma_i$. Also, I've specified that the graph should be unweighted (otherwise the assertion is false).

4 Undergraduate Problems

1. Let G be the graph with vertex set $\{1, 2, \ldots, 2k\}$ and edge set

$$\{(i,j) : 1 \leq i < j \leq k\} \cup \{(i,j) : k+1 \leq i < j \leq 2k\} \cup \{(1, k+1)\}.$$

We call this the dumbell graph on $2k$ vertices. Let L be the Laplacian matrix of this graph, and let its eigenvalues be $0 = \gamma_1 < \gamma_2 \leq \cdots \leq \gamma_n$. Let $\mathcal{L}$ be the normalized Laplacian of this graph, and let its eigenvalues be $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$.

(a.) Prove that $\gamma_2 \leq 2/k$.

(b.) Prove that $\lambda_2 \leq 2/(k(k-1))$.

2. Let G be a connected unweighted graph. Let L be the Laplacian matrix of this graph, and let its eigenvalues be $0 = \gamma_1 < \gamma_2 \leq \cdots \leq \gamma_n$. Let $\mathcal{L}$ be the normalized Laplacian of this graph, and let its eigenvalues be $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$. Prove that for all i,

$$\gamma_i \geq \lambda_i.$$

3. Let $G = (V, E)$ be a connected, unweighted, undirected graph with n vertices. Let s and t be distinct vertices in V. Prove that the effective resistance between s and t is at most $n - 1$.

   Hint: First, consider the case in which the graph just consists of a path from s to t. Then, use Rayleigh's Monotonicity Theorem.

4. The Sherman-Morrison Formula from linear algebra tells us that if A is a symmetric matrix and u is a column vector in the span of A, then

$$\left(A + uu^T\right)^+ = A^+ - \frac{A^+ uu^T A^+}{1 + u^T A^+ u},$$

   where A^+ denotes the pseudo-inverse of A. (If you want, you could check this by multiplying through, keeping in mind the fact that the pseduo-inverse is juat like the inverse on the span on A).

   Use the Sherman-Morrison Formula to prove Rayleigh's Monotonicity Theorem.

   Hint: $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}$.

5. Exhibit a graph $G = (V, E)$ for which there are nodes a, b, c in V such that

   - The distance from a to b is less than the distance from a to c, but
   - $\mathrm{R}_{\mathrm{eff}}(a, c) < \mathrm{R}_{\mathrm{eff}}(a, b)$.

5 Gradute Problem

Consider the approach to finding cuts in graphs suggested by Grady: to separate s from t, let $\boldsymbol{v}$ be the potential of the unit electrical flow from s to t, and take a cut of the form

$$S = \{u : \boldsymbol{v}(u) \leq x\},$$

for some threshold x.

Your goal in this problem is to show that this approach can fail to find even reasonable approximations of the lowest-conductance cut separating s from t. For every constant k, find a graph in whichthe lowest conductance cut separating s from t is at least k times smaller than the conductance of every cut produced by Grady's algorithm. Clearly, k will be a function of the number of vertices, n, in your graph. Try to make k as large a function of n as possible.

6 Experimental Problem

Cut the graph you gathered last week. Use either a cut from an eigenvector, or use Grady's approach. What is the least conductance cut you can find? Does it have any meaning in your graph?

Submit all code.

For easy access to the numerical routines you will need, I suggest either using Matlab or numerical packages available in Python. Note that it can take a while to get any package working.

If you are going to try the spectral approach, you could try to get the vectors corresponding to the random walk. One way to do this is to keep multiplying be the random walk matrix (this is easy to implement), and to keep your vector orthongonal to the all-1's matrix. This is easy to implement in any language, but can be slow. Feel free to discuss this with me or Jae Oh.