2024-Feb-19 Harmonic Functions. For $G=(V,E,w)$ connected. $B \subset V$, boundary

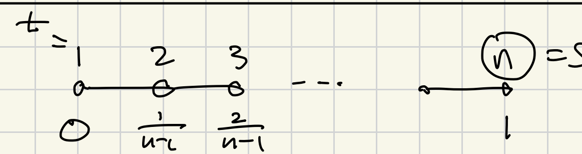
$S = V - B$. $x : V \to \mathbb{R}$ is <u>harmonic</u> on S if $\forall a \in S$ $x(a) = \frac{1}{d(a)} \sum_{b \sim a} w_{a,b}\, x(b)$

<u>Example</u> pick $s,t \in V$, set $B = \{s,t\}$. Consider random walk that stops when hits s or t.

$x(a) = \Pr[\text{walk that starts at } a \text{ stops at } s]$ $x(s)=1$ $x(t)=0$. x is harmonic on S

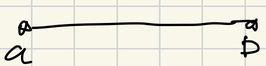
For $a \notin B$, $x(a) = \Pr[\text{stops at } s \mid \text{at } a] = \sum_{b \sim a} [\text{moves from } a \text{ to } b] \cdot \Pr[\text{stops at } s \mid \text{at } b] = \sum_{b \sim a} \frac{w_{a,b}}{d(a)} x(b)$

Ex. Consider a path P_n. $t=1$, $s=n$ $x(a) = \frac{a-1}{n-1}$



Spring Networks. Each edge is an ideal linear spring with spring constant = weight

Fix position a boundary. let Hooke's Law determine positions of the others.



force exerted on a is $w_{a,b}(x(b) - x(a))$. At equilibrium forces sum to 0

$\forall a \in S$ $\sum_{b \sim a} (x(b) - x(a)) w_{a,b} = 0 \iff \sum_{b \sim a} w_{a,b}\, x(b) = x(a)\, d(a) \iff$ harmonic at a

Path: if set $x(1) = 1$, $x(n) = n$, $B = \{1, n\}$, get $x(a) = a$ for all a.

Solutions: existence, uniqueness, and how to find them.

$a \in S \to d(a)x(a) - \sum_{b \sim a} w_{a,b} x(b) = 0$ is $\delta_a^T L x = 0$, a row of L for each $a \in S$

move $b \in B$ to rhs $d(a)x(a) - \sum_{\substack{b \sim a \\ b \in S}} w_{a,b} x(b) = \sum_{\substack{b \sim a \\ b \in B}} x(b)$ becomes $L(S,S)x(S) = M(S,B)x(B)$

$$x(S) = L(S,S)^{-1} M(S,B) x(B)$$

Need to show $L(S,S)^{-1}$ exists. True for connected

and $B \neq \emptyset$

$L(S,S) = L_{G(S)} + H_S$ where H_S diagonal $H_S(a,a) = \sum_{\substack{b \sim a \\ b \in B}} w_{a,b}$

lem: Let G be connected, H non-neg diagonal, $H \neq 0$. Then $L_G + H$ is pos definite

proof: wts $\forall x$ $x^T(L_G + H)x > 0$. $\forall x$ $x^T L_G x \geq 0$, $x^T H x \geq 0$

If x is non-constant, $x^T L_G x > 0$ (because connected)

If $x = c\mathbb{1}$, $x^T H x = c^2 \sum_a H(a,a) > 0$. Either way is > 0

Almost proves $L(S,S)$ pos def, but $G(S)$ could be disconnected.

$$\begin{bmatrix} L(S_1,S_1) & 0 & 0 \\ 0 & L(S_2,S_2) & \\ 0 & 0 & \end{bmatrix}$$

<u>Thm</u>' If G is connected, $B \neq \emptyset$, $S = V - B$, then $L(S,S)$ is pos. def.

<u>proof</u>

let $S_1, \ldots, S_k$ be connected components of $G(S)$

Just need to show each $L(S_i,S_i)$ is pos def. $G(S_i)$ is connected, by definition

$L(S_i,S_i) = L_{G(S_i)} + H_i$, where $H_i(a,a) = \sum_{\substack{b \sim a \\ b \notin S_i}} w_{a,b}$. Not identically 0 because G connected $\Rightarrow$ edge on boundary of S_i

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<u>Energy</u> Potential energy in spring of length l with constant w is $\frac{1}{2} w l^2$

So, total energy in network is $\frac{1}{2} \sum_{(a,b) \in E} w_{a,b} (x(a) - x(b))^2 = \frac{1}{2} x^T L x$

Physics: energy minimized at equilibrium. So, takes $\dfrac{\partial \frac{1}{2} x^T L x}{\partial x(a)} = 0$

$$\frac{1}{2} \frac{\partial x^T L x}{\partial x(a)} = \frac{1}{2} \sum_{b \sim a} 2(x(a) - x(b)) w_{a,b} = 0 \iff x \text{ harmonic at } a$$

Resistor Networks

Resistance of edge a,b is $r_{a,b} = \frac{1}{w_{a,b}}$

Voltages on vertices. Current on edges. Ohm's law $V = IR$ — current, resistance, volt diff

$i(a,b)$ = flow from a to b $= -i(b,a)$

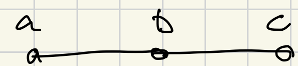
$v(a) - v(b) = i(a,b)\, r_{a,b}$ current from high to low $\qquad i(a,b) = w_{a,b}(v(a) - v(b))$

U = signed edge-vertex adj matrix is $E \times V$ $\qquad U((a,b),c) = \begin{cases} 1 & a=c \\ -1 & b=c \\ 0 & \text{o.w.} \end{cases}$

pick an arbitrary orientation for each edge.

$W = E \times E$ diagonal edge weight matrix. Now set $i = WUv$

$i_{ext} \in \mathbb{R}^V \triangleq$ current entering. $\qquad i_{ext}(a) = \sum_{b \sim a} i(a,b)$ no current stored. $\quad i_{ext} = U^T i$



$$U = \begin{array}{c|ccc} & a & b & c \\ \hline a,b & 1 & -1 & 0 \\ b,c & 0 & 1 & -1 \end{array}$$

$L = U^T W U \qquad i_{ext} = Lv$

$$= \sum_{(a,b) \in E} w_{a,b} (\delta_a - \delta_b)(\delta_a - \delta_b)^T$$

$B = \{a : i_{ext}(a) \neq 0\}$

v:	-1	1	0
Uv:	-2	1	
i:	-2	1	
$U^T i$:	-2	3	-1 $= i_{ext}$

For $a \in S$ $\quad i_{ext}(a) = 0 \quad \rightarrow \quad \delta_a^T L v = 0 \quad$ so $\quad v$ harmonic at a

$$\rightarrow \sum_{b \sim a} w_{a,b} (v(a) - v(b)) = 0. \quad \text{As} \quad i(a,b) = w_{a,b}(v(a) - v(b)) \longrightarrow \sum_{b \sim a} i(a,b) = 0$$

Given i_{ext}, how solve for v? $L^{-1} i_{ext}$? Need $\mathbb{1}^T i_{ext} = 0$. Use $L^+ i_{ext}$

the pseudo-inverse. is inverse on span of L. $L^+ L = L L^+ = \Pi$ = projection onto span(L)

If connected, $\Pi = \frac{1}{n} L_{K_n}$. If $L = \sum \lambda_i \psi_i \psi_i^T$ $\quad L^+ = \sum_{i: \lambda_i \neq 0} \frac{1}{\lambda_i} \psi_i \psi_i^T$
