Recall \( E^B \)
\[
L = B^T W B \quad i = \omega B n \quad i_{\text{ext}} = L n
\]
\[
N = L^+ i_{\text{ext}}.
\]

**Effective resistance**
\[
U = I R \quad R = \frac{V}{I}
\]
\[
= \frac{\text{voltage difference}}{\text{current flow}}.
\]

Consider \( i_{\text{ext}} = \delta a - \delta b \) flows 1 from \( a \) to \( b \) in graph
\[
N = L^+ i_{\text{ext}} = L^+ (\delta a - \delta b)
\]

difference \( = N(a) - N(b) = (\delta a - \delta b)^T U \)

Refl \( (a, b) \) \( = (\delta a - \delta b)^T L^+ (\delta a - \delta b) \)

Will see later that is a distance. For now,

\[
\text{let } L^{+i_2} = \left( L^+ \right)^{i_2}
\]

Every psd matrix \( A \) has \( A^{i_2} \) s.t. \( A^{i_2} A^{i_2} = A \)

\[
\text{if } A = \sum \lambda_i \psi_i \psi_i^T \quad A^{i_2} = \sum \lambda_i^{i_2} \psi_i \psi_i^T
\]

\[
\text{Reff}(a, b) = (\delta a - \delta b)^T L^{+i_2} \cdot L^{+i_2} (\delta a - \delta b)
\]
\[
= \| L^{+i_2} (\delta a - \delta b) \|^2_2
\]
\[
= \| L^{+i_2} \delta a - L^{+i_2} \delta b \|^2_2
\]
By energy minimization.
Recall in spring of constant $w$, $E = \frac{1}{2}wL^2$
when stretched to length $L$.
So, let's stretch to length $L$,
measure minimum energy, and double to set effective spring constant

$E(x) = \frac{1}{2} \sum_{a<b} w_{ab} (x(a) - x(b))^2$

Let's fix $x(s) = 1$, $x(t) = 0$
And want harmonic elsewhere.

Set $\gamma = \frac{L^+ (\delta_s - \delta_t)}{Reff(s,t)}$

Gives $y(s) - y(t) = \left( \begin{array}{c} \delta_s - \delta_t \end{array} \right)^T L^+ (\delta_s - \delta_t) = 1$

Could shift by $x = \gamma - 1 \gamma(t)$
Now $x(t) = 0$, $x(s) = 1$
Harmonic on $U - \delta s \delta t$ => minimizes energy
\[ E(x) = \frac{1}{2} x^T L x = \frac{1}{2} y^T y \]
\[
= \frac{1}{2} \frac{1}{\text{Reff}(s,t)^2} (d_s - d_t)^T L^T L L^T (d_s - d_t) \\
= \frac{1}{2} \frac{1}{\text{Reff}(s,t)^2} (d_s - d_t)^T L^T L (d_s - d_t) \\
= \frac{1}{2} \frac{1}{\text{Reff}(s,t)}
\]

So, effective spring constant $= \frac{1}{\text{Reff}(s,t)}$

**Classic examples**

Path with $n$ vertices, edges of resistance $T_1 \cdots T_{n-1}$

\[ \text{Reff}(1,n) = T_1 + \cdots + T_{n-1} \]

Proof set \[ N(1) = T_1 + \cdots + T_{n-1} \quad V(1) = 0, \quad V(n) = T_1 + \cdots + T_{n-1} \]

Current over edge $(c_i, a)$ is \[ \frac{V(a) - V(c_i)}{T_{a - c_i}} = -\frac{T_{a - c_i}}{T_{a - c_i}} \]

So, corresponds to a flow of value 1 from $a$ to $c_i$.

Parallel edges

Claim \[ \text{Reff}(s,t) = \frac{1}{T_1 + \cdots + T_n} \]
proof \[ P = \frac{U}{t} \quad \text{set} \quad U(0)=1 \quad U(t)=0 \]

flow on edge \( i \) is \[ \frac{(U(0)-U(t))}{t_i} = \frac{1}{t_i} \]

So total flow \[ \sum_i \frac{1}{t_i} \]

\[ \Rightarrow \text{Reff}(S,t) = \frac{1}{\sum_i \frac{1}{t_i}} \]

If view formula as \( i = (U(0)-U(t))\omega_{st} \)

\( \omega_{st} = \frac{1}{t_i} \), then add weights of parallel edges.

---

Equivalent networks. Given \( B \), want matrix \( L_B \)

s.t. \( \Delta B = L_B v(B) \) when \( \nu \) hormone on \( S = V-B \).

To do it slowly, first consider \( B = \{2, \ldots, n\} \), \( S = \{1\} \)

Let \( N = \{a: a \sim 1\} \).

Want to compute \( L_B \) given \( v(B) \) and \( u(i) = \frac{1}{d(i)} \sum_{a \sim i} \omega_{ai} \cdot v(a) \)

Substitute for \( u(i) \) in

\[ \Delta a = d(a) u(a) - \sum_{b \sim a} \omega_{ab} u(b) \] when \( i \neq q \)

no change at \( a+1 \)
For $a \sim 1$ becomes

\[
d(a) u(a) - \sum_{b \sim q} w_{ab} u(b) - w_{a1} \sum_{c=1}^{b+1} w_{1c} u(c)
\]

\[
= u(a) \left[ d(a) - \frac{w_{12}}{d(1)} \right] - \sum_{c=1}^{b+1} \frac{w_{a1} w_{1c}}{d(1)} u(c) - \sum_{b \sim q} w_{a1b} u(b)
\]

Claim is result of elimination on row(col 1 and 1) is a Laplacian equation on $B$.

We removed node 1 and attached edges, and
put back a clique on nbrs,
where for $a, c \sim 1$ have edge of wt $\frac{w_{a1} w_{1c}}{d(1)}$

Proof is Laplacian

1. is symmetric: same chain to $\text{row}(c)$ in $a$

2. off diagonal terms negative

3. sum of coefficients is 0, as

\[
W_{1a} - \frac{W_{1a}^2}{d(1)} - \sum_{c=1}^{b+1} \frac{W_{1c} W_{1q}}{d(1)} = W_{1a} - \frac{W_{1q}}{d(1)} \sum_{c=1}^{b+1} W_{1c} = 0
\]
In energy term,
Compute $L_B$ so that
\[ U(B)^T L_B U(B) = \left( \frac{1}{\alpha_i} \sum_{a=1}^{\alpha_i} \omega_{ia} v(a) \right)^T L \left( \frac{1}{\alpha_i} \sum_{a=1}^{\alpha_i} \omega_{ia} v(a) \right) U(B) \]

idea: \[ \frac{1}{\alpha_i} \sum_{a=1}^{\alpha_i} \omega_{ia} v(a) = - \frac{L(1,1) U(B)}{L(1,1)} \]

Substituting this in yields
\[ U(B)^T \left[ L(B,B) - \frac{L(B,1) L(1,1)}{L(1,1)} \right] U(B) \]

To check Laplacian, note: only decrease entries,
and $L_B I_B = 0$ provf gives $\n(1) = 1$, so $\n^T u = 0$

Or, \[ I_B^T L(B,B) I_B = d(1) \] and \[ L(1,1) I_B = d(1) \]
\[ L(1,1) = d(1). \]

Is what get by GE on row/row.
Eliminating many vertices at once:

Does not depend on order!

To eliminate entries in row \( a \in B \) and cols in \( S \)
using rows in \( S \), mult by coeffs \( c \) so that

\[
L(a_i S) - c L(S_i S) = 0
\]

So,

\[
c = L(a_i S) L(S_i S)^{-1}
\]

giving \( L_i(a_i) = L(a_i) - L(a_i) L(S_i S)^{-1} L(S_i B) \)

restricting to rows and cols in \( B \) we get

\[
L_B(B_i B) = L(B_i B) - L(B_i S) L(S_i S)^{-1} L(S_i B)
\]

is Schur complement with respect to \( S \)

or onto \( B \)

To show equiv of harmonic on \( S \):

\[
\text{harmonize} \Rightarrow L(S_i S) v(S) + L(S_i B) v(B) = 0
\]

\[
v(S) = - L(S_i S)^{-1} L(S_i B) v(B)
\]

\[
\text{ext}(B) = L(B_i S) v(S) + L(B_i B) v(B)
\]

\[
= \left[ L(B_i B) - L(B_i S) L(S_i S)^{-1} L(S_i B) \right] v_B
\]
If \( B = \{ s, t \} \) get down to one edge, 
whose weight is \( \frac{1}{Reff(s,t)} \).

In particular, for \( s, t \in B \),
\[
(\overrightarrow{d_s - d_t})^T L_B (\overrightarrow{d_s - d_t}) = (\overrightarrow{d_s - d_t})^T L^T (\overrightarrow{d_s - d_t})
\]

Is how GE wants to solve \( i_{ext} = Lu \)
Order vertices \( l, \ldots, n \)

Construct \( L_{\{a,\ldots,n\}} \) for each \( a \).

Given \( i_{ext(a)} \) and \( N(a) : l, \ldots, N(a) \)

solve for \( N(a) \).

If \( i_{ext(a) = 0} \), \( N(a) = \sum_{b \in a} N(b) \)

0(w) need to account for \( i_{ext(a)} \)

Reff as distance:
assert \( \forall a, b, c \quad Reff(a,b) + Reff(b,c) \geq Reff(a,c) \)

Only need to prove for 3-node graphs.