2025-Feb-24. $G=(V,E,w)$ $B\subset V$ $S=V-B$ x harmonic on S if $\forall a\in S$ $x(a)=\frac{1}{d(a)}\sum_{b\sim a} w_{a,b}\,x(b)$

Spring network: if fix $x(b)$ for $b\in B$, $x(S)$ determined by harmonic: $x(S)=L(S,S)^{-1}M(S,B)x(B)$

Minimizes energy $\mathcal{E}(x)\triangleq\frac{1}{2}x^TLx$ given conditions on $x(B)$

Resistor network: if fix voltages v, $i_{ext}=Lv$. If fix i_{ext}, $v=L^+i_{ext}$ ← flow in

$v(1)$ $v(0)$

$L=\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$ $v=\begin{pmatrix}1\\0\end{pmatrix}$ gives $i_{ext}=\begin{smallmatrix}1\\-1\end{smallmatrix}$

$1\uparrow$ $\downarrow -1$

A flow i is a potential flow if $i=WUv$ $i(a,b)=w_{a,b}(v(a)-v(b))$

harmonic at a if flow-in = flow out : $0=\sum_{b\sim a} i(a,b)=\sum_{b\sim a} w_{a,b}(v(a)-v(b))$

Effective resistance between a and b $V=IR$, $R=\frac{V}{I}$

$i_{ext}=\delta_a-\delta_b$ $v=L^+i_{ext}$ $v(a)-v(b)=(\delta_a-\delta_b)^TL^+(\delta_a-\delta_b)\triangleq R_{eff}(a,b)$

Spring with const w: $\mathcal{E}=\frac{1}{2}w\ell^2$ when stretch to length ℓ. Set $B=\{a,b\}$

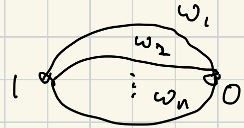
min energy by x harmonic on $S=V-B$. So, $x=L^+(\delta_a-\delta_b)$ is harmonic on S

$\mathcal{E}(x)=((\delta_a-\delta_b)^TL^+)L(L^+(\delta_a-\delta_b))=(\delta_a-\delta_b)^TL^+(\delta_a-\delta_b)$ = eff spring const

$\ell=(\delta_a-\delta_b)^Tx=(\delta_a-\delta_b)^TL^+(\delta_a-\delta_b)$ So $C_{eff}=\frac{1}{(\delta_a-\delta_b)^TL^+(\delta_a-\delta_b)}=\frac{1}{R_{eff}}$

Examples. Parallel

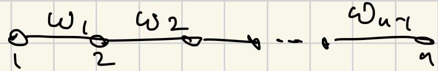
$w_i = 1/r_i$



flow is $w_1 + w_2 + \dots + w_n$

$$R_{eff} = \frac{1}{w_1 + \dots + w_n} = \frac{1}{1/r_1 + \dots + 1/r_n}$$

Series:



set $v(1) = 0$, $v(2) = \frac{1}{w_1}$, $v(i) = \frac{1}{w_1} + \dots + \frac{1}{w_{i-1}}$

$i(a, a+1) = w(a)\big(v(a) - v(a+1)\big) = \frac{-w_a}{w_a} = -1$ so flow is 1 from n to 1, harmonic

$$R = \frac{V}{I} = v(n) - v(1) = \frac{1}{w_1} + \dots + \frac{1}{w_{n-1}} = r_1 + \dots + r_n$$

Equivalent networks. Given B, want matrix $L_B \in \mathbb{R}^{B \times B}$ s.t. $i_{ext}(B) = L_B v(B)$

don't yet know is Laplacian

Consider $B = \{2, \dots, n\}$ $S = \{1\}$ let $A = \{a : (1,a) \in E\}$ nbrs

will have $v(1) = \frac{1}{d(1)} \sum_{c \in A} w_{1,c} v(c)$

For eqn $i_{ext}(a) = d(a)v(a) - \sum_{b \sim a} w_{a,b} v(b)$ when $a \neq 1$, $a \notin A$, no change to equation

For $a \in A$ $= d(a)v(a) - \sum_{\substack{b \sim a \\ b \neq 1}} w_{a,b} v(b) - \frac{w_{a,1}}{d(1)} \sum_{c \in A} w_{1,c} v(c)$

$$= v(a)\left[d(a) - \frac{w_{1,a}^2}{d(1)}\right] - \sum_{c \in A,\, c \neq a} \frac{w_{a,1}}{d(1)} w_{1,c} v(c) - \sum_{\substack{b \sim a \\ b \neq 1}} w_{a,b} v(b)$$

Claim: this system of equation is Laplacian (can write as $D-M$ $M \geq 0$, $(D-M)\mathbb{1}=\vec{0}$)

proof is symmetric: for $a, c \in C$, subtract $\frac{w_{a,1} w_{1,c}}{d(1)}$ from (a,c) entry

off-diagonals non-pos: row-sum for a changes by

$$+w_{1,a} - \frac{w_{1,a}^2}{d(1)} - \sum_{\substack{c \in A \\ c \neq a}} \frac{w_{a,1} w_{1,c}}{d(1)} = w_{1,a} - w_{1,a} \underbrace{\sum_{c \in a} \frac{w_{1,c}}{d(1)}}_{1} = 0$$

And, is what set in row a when do gauss elim with row 1:

$$L(a,\cdot) - \frac{L(a,1)}{d(1)} L(1,\cdot) \rightarrow$$

entry 1 is 0 because $L(1,1) = d(1)$

entry a goes to $d(a) - \frac{L(a,1)^2}{d(1)} = d(a) - \frac{w_{1,a}^2}{d(1)}$

entry c for $L(1,c) = -w_{1,c}$ goes to $L(a,c) - \frac{w_{a,1} w_{1,c}}{d(1)}$

Do in Spring network energy, with $B \subset V$, $S = V - B$, $x(B)$ fixed, x harmonic on S

$$x(S) = L(S,S)^{-1} M(S,B)\, x(B) = -L(S,S)^{-1} L(S,B)\, x(B) \quad \text{want } L_B \text{ so that } x(B)^T L_B\, x(B)$$

$$= \begin{pmatrix} x(S) \\ x(B) \end{pmatrix}^T \begin{pmatrix} L(S,S) & L(S,B) \\ L(B,S) & L(B,B) \end{pmatrix} \begin{pmatrix} x(S) \\ x(B) \end{pmatrix} = \Big(+ x(B)^T L(B,S) L(S,S)^{-1} \cancel{L(S,S) L(S,S)^{-1}} L(S,B)\, x(B)$$

$$\Big(-2\, x(B)^T L(B,S) L(S,S)^{-1} \Big) L(S,B)\, x(B) + x(B)^T L(B,B)\, x(B)$$

$$= x(B)^T L(B,B)\, x(B) - x(B)^T L(B,S) L(S,S)^{-1} L(S,B)\, x(B)$$

$$L_B = L(B,B) - L(B,S) L(S,S)^{-1} L(S,B)$$

To see is result of gauss elim of entries in S, note for $b \in B$,

to elim $L(b,S)$, add $c^T L(S,S)$ where $c^T L(S,S) = -L(b,S)$ so $c^T = -L(b,S) L(S,S)^{-1}$

so, change row b by adding $-L(b,S) L(S,S)^{-1} L(S,B)$

As c is unique, get same answer as if elim one vertex at a time.
To see preserves being Laplacian, note

$$L(B,B) - \frac{L(B,1) L(1,B)}{L(1,1)}$$ is what get after 1 step, and this is Laplacian

$\delta(a,b)$ is a distance if $\forall a,b,c$

$\delta(a,a) = 0$

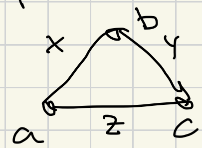
$\delta(a,b) \geq 0$

$\delta(a,b) = \delta(b,a)$

$\delta(a,c) \leq \delta(a,b) + \delta(b,c)$ $\triangle$

Thm Reff(a,b) is a distance:

proof. suffices to check $\triangle$ on 3-node graphs.



x, y, z = resistances

from a to c have z in parallel with x y

So, $\mathrm{Reff}(a,c) = \dfrac{1}{1/z + \frac{1}{x+y}} = \dfrac{zx+zy}{x+y+z}$

So, just need to show $\dfrac{zx+zy}{x+y+z} \leq \dfrac{\cancel{xy} + xz}{x+y+z} + \dfrac{\cancel{yx} + yz}{x+y+z}$ ✓

Tight case $z \to \infty$, $w_{a,c} = 0$