A $d$-regular graph $G$ is an $\varepsilon$-expander if $\mu_2 \leq \varepsilon d$ and $\mu_n \leq \varepsilon d$

Equivalent to $|\lambda_i - d| \leq \varepsilon d$ for $i \geq 2$, Cohnen

And $\|L_G - \frac{d}{n} \cdot K_n\| \leq \varepsilon d$

where $\|A\| = \max_x \frac{|Ax_i|}{\|x_i\|}$ for sym $A$, $\|A\| = \max (\text{eigen} (A) |$

Recall for $i \geq 2$, $\lambda_i (K_n) = n$, so $\lambda_i (\frac{d}{n} K_n) = d$

How small $\varepsilon$?

Ramanujan: $\varepsilon = \frac{2\sqrt{d} - 1}{d}$

Today: $\varepsilon \leq \frac{2\sqrt{d} - 1 - c}{d}, c > 0$, impossible for larger $d$ fixed.

For $A, B \subseteq V$, $A \cap B = \emptyset$

$E(A, B) = \{(a, b) \in E : a \in A, b \in B\}$ disjoint

If $G$ is $d$-regular, pick $A$ and $B$ at random, so $E(A, B)$

$\Pr[\{a \in A, b \in B\}] = \alpha_B$

$\Pr[E(A, B)] = 2\alpha_B$

$|E(A, B)| = \frac{dn}{2} - 2\alpha_B = d\alpha_B n$
For $K_n$, $(A=Kn, \ |B|=p^n$)

$$E(A,B) = C(\alpha)(\beta^n) = \alpha \beta \mu^2, \ \text{with} \ \mu = p^{-1}$$

Then $4A \cap B$ s.t. $A \cap B = \emptyset$,

$$|E(A,B) - d\alpha \beta \mu| \leq 2d \sqrt{\alpha - \alpha^2 - \beta - \beta^2}$$

is small error when $\alpha \beta > \varepsilon$

**Proof**

$$1^T_A (L \alpha - L \hat{\alpha}) 1_B = -E(A,B)$$

Let $(t = \frac{1}{n} K_n, \ \ 1^T_A L \alpha 1_B = -\frac{1}{n} |A| \cdot |B| = d \alpha \beta \mu$

$$|1^T_A (L \alpha - L \hat{\alpha}) 1_B| = |E(A,B) - d \alpha \beta \mu|$$

$$\leq \|1_A\| \cdot \|L \alpha - L \hat{\alpha}\| 1_B| \text{ is Cauchy - Schwartz}$$

$$\leq \|1_A\| \cdot \|L \alpha - L \hat{\alpha}\| \|1_B\| \text{ by norm}$$

$$\leq \sqrt{\mu \sqrt{\beta \mu}} \ \text{and} \ \| \| = d \sqrt{\mu \sqrt{\beta \mu}}$$

To improve, let $x_A = 1_A - \alpha \hat{\alpha} \ \ x_B = 1_B - \beta \hat{\beta}$

Now,

$$1^T_A (L \alpha - L \hat{\alpha}) 1_B = x_A^T (L \alpha - L \hat{\alpha}) x_B$$

$$\|x_A\| = \sqrt{\mu (\alpha - \alpha^2)} \ \ \|x_B\| = \sqrt{\mu (\beta - \beta^2)}$$

Gives the result.
For $A \subseteq V$, $N(A) = \{b : \exists a \in A, (a,b) \in E\}$

Tanner's Theorem

For $|A| = \alpha n$, $(NCA) = \delta n, \delta = \frac{\lambda}{\varepsilon^2(1-\alpha) + \alpha}$

For $\alpha$ small, think of as $\frac{\alpha}{\varepsilon^2}$ for $\varepsilon = \frac{\sqrt{\beta}}{d}$

is like $\frac{\lambda}{\varepsilon^2(1-\alpha) + \alpha}$

Proof: $T = U - N(A)$ so no edges between $T$ and $A$

$\beta n = |T|, \beta = 1 - \delta$

$\alpha \beta \delta n \leq 2 d n \sqrt{\alpha(1-\alpha) \beta(1-\beta)}$

$\sum \alpha \beta \leq 2 \sqrt{(\alpha)(1-\alpha)}$

$\alpha \beta \leq \varepsilon^2 (1-\alpha)(1-\beta)$

$\frac{\beta}{1-\beta} \leq \varepsilon^2 (1-\alpha) \frac{\alpha}{1-\alpha}$

$\frac{1-\delta}{\delta} \leq \varepsilon^2 (1-\alpha) \frac{\alpha}{1-\alpha}$

$\frac{1}{\delta} \geq \varepsilon^2 (1-\alpha) + 1$

$\gamma = \frac{\lambda}{\varepsilon^2 (1-\alpha) + \alpha}$
How small can $\epsilon$ be?

For $|d|=1$, $d=\frac{1}{k}$, $|\mathcal{N}(d)|=d$, $x=\frac{1}{d}$, $\epsilon \geq \frac{1}{\sqrt{d}}$

will show for large graphs is an $\times 1$ s.t.

$$\frac{x^T L x}{x^T x} \geq d - 2\sqrt{d} + \text{ small}$$

Let $a, b \in V$, $\mathcal{N}(a) \cap \mathcal{N}(b) = \emptyset$

and no edges between $\mathcal{N}(a)$ and $\mathcal{N}(b)$

$$x(a) = \begin{cases} 1 & \text{ if } a = b \\ -1 & \text{ if } a \neq b \\ \frac{1}{\sqrt{d}} & c \in \mathcal{N}(a) \\ -\frac{1}{\sqrt{d}} & c \in \mathcal{N}(b) \end{cases}$$

$x \perp 1$, so $x_2 \leq \frac{x^T L x}{x^T x}$

$$= \frac{d(1 - \frac{1}{\sqrt{d}})^2 + d (d-1) \left( \frac{1}{\sqrt{d}} \right)^2 + \text{ same}}{(1 + d(\frac{1}{\sqrt{d}})^2) + \text{ same}}$$

$$= \frac{d - 2\sqrt{d} + 1 + d - 1}{2} = d - \sqrt{d}$$

we already knew that!
Improve by basing arcs on far apart edges

Assume are edges \((a_0, a_i), (b_0, b_i)\) at distance \(2k+2\)

If all fixed, \(n \times k\), this happens:

\[
\lambda_2 \leq d - 2J_{d-1} + \frac{2J_{d-1} - 1}{k+1}
\]

\(A_0 = \{ a_0, a_i \} \quad A_i = \{ a : \text{dist}(a_0, a) = i \} \)

\(B_0 = \{ b_0, b_i \} \quad B_i = \{ a : \text{dist}(a, b_0) = i \} \)

\(A_0, \ldots, A_k \) disjoint, \(B_0, \ldots, B_k \) disjoint

no edges from \(A_i \) to \(B_j\), any \(i, j \in k\)

\[
\begin{align*}
\chi(A_0) &= 1 \\
\chi(A_i) &= (d-1)^{-i/2} \\
\chi(B_0) &= -\beta \\
\chi(B_i) &= -\beta (d-1)^{i/2}
\end{align*}
\]

choose \( \beta \) s.t. \( A^T x = 0 \)
\[
\frac{x^T L x}{x^T x} = \frac{A \cdot \text{edges} + B \cdot \text{edges}}{A \cdot \text{verts} + B \cdot \text{verts}} \leq \max \left( \frac{A \cdot \text{edges}}{A \cdot \text{verts}}, \frac{B \cdot \text{edges}}{B \cdot \text{verts}} \right)
\]

\[
A \cdot \text{verts} = \sum_{i=0}^{k} |A_i| \cdot (d-1)
\]

\[
A \cdot \text{edges} \leq \sum_{i=0}^{k-1} \frac{|A_i|}{(d-1)^i} \left( d-2jd-1 \right) + \frac{|A_{k-1}|}{(d-1)^k} \left( d-1 \right)
\]

\[
= \sum_{i=0}^{k-1} \frac{|A_i|}{(d-1)^i} \left( d-2jd-1 \right) + \frac{|A_{k-1}|}{(d-1)^k} \left( d-1 \right)
\]

\[
\leq (d-2jd-1) A \cdot \text{verts}
\]

\[
\Rightarrow \frac{|A_{k-1}|}{(d-1)^k} \leq (d-1) ^{-k-i} \cdot \sum_{i=0}^{k} \frac{|A_i|}{(d-1)^i}
\]

\[
\frac{|A_{k-1}|}{(d-1)^k} \left( 2jd-1 \right) \leq \frac{1}{k+1} \left( 2jd-1 \right) \cdot \sum_{i=0}^{k} \frac{|A_i|}{(d-1)^i}
\]

\[
\leq A \cdot \text{verts}
\]