Spectral and Algebraic Graph Theory 462/562
(formerly grad course numbered AMTH 561/CPSC 662)
Spectral = eig vals and eig vecs

Grad/grad => I assume less.
  Grads get extra homework problems
  More applied ICS focus
  But still very mathy

Can find lecture notes from previous years.
Writing a book now. The book will have
details I cannot cover in class.

I will distribute my handwritten notes, when I
have them.

Assignments: 5-6. Probably 6. Can work in small groups
(for now). No tests or exam.

Occasional recommended exercises, especially for
this lecture.

Prereqs: linear algebra, graph theory, some probability.
  proof-based exposition, endurance

Today: Intro, overview, a proof or two.
  Get used to my notation.
Please interrupt when necessary.
Graphs. \( G = (V, E) \) \( E \) is set of pairs of \( V \).
Write edges as \((a, b)\), although \((a, b)\) would be better.
\((a, b) = (b, a)\)
Undirected.
No self-loops or multi-edges.
Usually.

Might be weights on edges. If so, almost always positive.

Sources: social networks, comm. networks, etc.
abstract like
path on \( n \) vertices. \( U = \{1, \ldots, n\} \)
\( E = \{(a, a+1) \mid 1 \leq a < n\} \)
ring: path plus edge \((1, n)\)
hypercube: \([01]^d\) \[ U = \{0, 1\}^d \]
\( (a, b) \in E \) if \( |\{i : a(i) + b(i)\}| = 1 \)
random.
edge \((a, b)\) appears with probability \( p \),
independently chosen.
Matrices for graphs.

Adjacency. \( M \) rows/cols labeled by \( V \).
\[
M(a,b) = \begin{cases} 1 & (a,b) \in E \\ 0 & \text{o.w.} \end{cases}
\]

Is using matrix as a spreadsheet.

Very surprising eigenvalues or eigenvectors should matter.

Diffusion Operator / walk Matrix.

Let \( D \) = diagonal matrix of degrees.
\[
d = M \cdot 1 \quad D = \text{diag}(d)
\]
\[
W = D^{-1} M
\]

Let \( p \in \mathbb{R}^V \) be a vector of \( p(a) \) = amount of stuff at vertex \( a \).

If stuff at \( a \) moves to neighbors of \( a \), evenly, then new distribution of stuff is
\[
p' = p' D^{-1} M
\]

Total amount of stuff, \( p' \cdot 1 \), is conserved because
\[
p' \cdot 1 = p' D^{-1} M 1 = p' D^{-1} d = p \cdot 1
\]

Expect spectra of \( W \) to matter.
Laplacian \[ L = D - M \]

defines a natural quadratic form:

for \( x \in \mathbb{R}^n \), \( x^T L x = \sum_{i,j} (x(i) - x(j))^2 \)

\( C(i,j) \in \mathbb{E} \)

\[ \text{e.g.} \quad \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 3 \end{array} \quad \begin{bmatrix} x^T L x \end{bmatrix} = 1^2 + 2^2 = 5 \]

**Spectral Theory**

\( \psi \) is an eigenvector of \( M \) of eigenvalue \( \lambda \) if

\[ M \psi = \lambda \psi \]

**Theorem** Every real symmetric \( n \times n \) matrix \( M \) has \( n \) real eigenvalues \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \) and \( n \) orthonormal eigenvectors \( \psi_1, \psi_2, \ldots, \psi_n \)

\[
\begin{pmatrix}
\psi_1^T \\
\psi_2 \\
\vdots \\
\psi_n
\end{pmatrix} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

s.t. \[ M \psi_i = \mu_i \psi_i \]

Note: \( \psi_1, \psi_2 \) not uniquely defined.

**Examples in Jupiter**
Course topics

Graph structure: cuts, coloring, indep sets, partitioning, local partitioning

The zoo: fundamental examples
Estimating eigenvalues
Random walks
Physical models: springs & resistors
Effective resistance and elimination
Expanders - extremal combinatorics
  relation to codes and pseudo-randomness
Sparsification
Solving Laplacian equations
  and computing eigenvectors.
**Def** The Rayleigh Quotient of \( x \) with respect to \( M \) is 
\[
\frac{x^T M x}{x^T x}
\]

**Theorem** If \( M \) is symmetric and \( x \) maximizes \( \frac{x^T M x}{x^T x} \) then \( Mx = \mu_i x \).

Will expand \( x = \sum_{i=1}^{n} c_i \psi_i \), where \( c_i = \psi_i^T x \)

Why? Well \( x = \sum x(i) \delta_i \), where \( \delta_i \) is eigenvector
in direction \( i \),
and \( x(i) = \delta_i^T x \).

And, \( \sum_i c_i \psi_i = \sum_i \psi_i \cdot \psi_i^T x = \left( \sum_i \psi_i \psi_i^T \right) x = \mathbf{I} x = x \)

**Claim** \( x^T M x = \sum_i c_i^2 \mu_i \), where \( c_i = \psi_i^T x \)

**Proof**
\[
x^T M x = \left( \sum_i c_i \psi_i \right)^T M \left( \sum_j c_j \psi_j \right)
\]
\[
= \left( \sum_i c_i \psi_i \right)^T \left( \sum_j c_j \psi_j \right)
\]
\[
= \sum_i c_i^2 \mu_i \]
\begin{align*}
\quad \text{as } \quad \Psi_i^T \Psi_j &= \begin{cases} 
1 & i=j \\
0 & \text{o.w.} \end{cases} \\
\quad = \sum_i C_i^2 \lambda_i \\
\end{align*}

\underline{Proof of Theorem} \quad \text{for all } x

\begin{align*}
\frac{x^T M x}{x^T x} = \frac{\sum_i C_i^2 \mu_i}{\sum_i C_i^2} &\leq \frac{\sum_i C_i^2 \mu_1}{\sum_i C_i^2} \\
\quad \text{as } C_i^2 &\geq 0 \text{ and } \mu_1 \geq \mu_i, \quad \forall i \\
\quad = \mu_1 \\
\end{align*}

and equality only holds if \( C_i^2 = 0 \) for \( \mu_i < \mu_1 \).