Every graph has an ε-approximation with $\approx \frac{4n}{\varepsilon^2}$ edges.

Ave degree is $8/\varepsilon^2$,

whereas Ramanujan bound is d-regular, $\varepsilon \sim \frac{2}{\sqrt{d}}$

which is degree $\approx 4/\varepsilon^2$.

so, looses a factor of 2

Last lecture wrote $L_G = \sum_{a \sim b} w_{a,b} L_{a,b}$

and observe that $L_H = \sum_{a \sim b} u_{a,b} L_{a,b}$ is ε-approx of L_G

if $\sum_{a \sim b} u_{a,b} L_G^{+/2} L_{a,b} L_G^{+/2}$ is ε-approx of $\Pi = \frac{1}{n} L_{K_n} = L_G^{+} L_G$

<u>Def</u> $v_{a,b} = \sqrt{w_{a,b}}\, L_G^{+/2} (\delta_a - \delta_b)$

so, $\Pi = \sum_{a \sim b} v_{a,b} v_{a,b}^T$

Will build a sparsifier by choosing $s: E \to \mathbb{R}_{\geq 0}$

s.t. $\sum s_{a,b} v_{a,b} v_{a,b}^T$ is ε-approx of Π

and at most $\approx \frac{4n}{\varepsilon^2}$ entries of s are nonzero.

so, $u_{a,b} = s_{a,b} \cdot w_{a,b}$ above ..

After restricting to span of Π, can rewrite problem as:

Given $\sum v_i v_i^T = I$,

find s_i most zero s.t.

$(1-\varepsilon) I \preceq \sum s_i v_i v_i^T \preceq (1+\varepsilon) I$

<u>Thm</u> Can find s with at most $\lceil n/\varepsilon^2 \rceil$ nonzero entries

s.t. $(1-\varepsilon)^2 I \preceq \sum s_i v_i v_i^T \preceq (1+\varepsilon)^2 I$

Today, we prove $\text{eigs}\left(\sum s_i v_i v_i^T \right) \in [n, 13n]$

at most $6n$ nonzero entries.

Then, rescale.

<u>Background</u> If $\sum v_i v_i^T = I$,

$\sum_i v_i^T M v_i = Tr(M)$.

<u>proof</u> $v_i^T M v_i = Tr[v_i v_i^T M]$

So $\sum_i v_i^T M v_i = \sum_i Tr[v_i v_i^T M] = Tr\left[\sum_i v_i v_i^T M\right]$

$= Tr[IM] = Tr[M]$

Sherman-Morrison For A nonsingular, vector v, real c

$$(A - c v v^T)^{-1} = A^{-1} + c \frac{A^{-1} v v^T A^{-1}}{1 - c v^T A^{-1} v}$$

proof substitution.

Iterative alg. Start with $A = 0$, $s_i = 0, \forall i$.

At each step adjust one entry of s,

add one vector to A.

keep eigvals of A between l and u.

at each step, $l \leftarrow l + \delta_L = l + \frac{1}{3}$

$u \leftarrow u + \delta_u = u + 2$.

Start with $l_0 = -n$, $u_0 = n$

After $6n$ steps, $l = n$, $u = 13n$

Measure $\Phi^u(A) = \sum_{i=1}^{n} \frac{1}{u - \lambda_i}$ λ_i eigvals of A

$$= \mathrm{Tr}\left((uI - A)^{-1} \right)$$

For $u > \lambda_1, \dots, \lambda_n$ $\rightarrow \infty$ as $u \rightarrow \lambda_{max}$

If $u > \lambda_{max}$, $u - \lambda_{max} \geq \frac{1}{\Phi^u(A)}$

Decreasing: $\phi^{u+\delta}(A) < \phi^{u}(A)$ for $\delta > 0$

Also convex

Initially $\phi^{u_0}(A) = \sum_i \frac{1}{n-0} = 1$

For lower side, track $\Phi_\ell(A) = \sum \frac{1}{\lambda_i - \ell} = \mathrm{Tr}\left((A-\ell I)^{-1}\right)$

$$\lambda_{min} \geq \ell + \frac{1}{\Phi_\ell(A)}$$

initially $\Phi_{\ell_0}(A) = \Phi_{-n}(0) = 1$

Claim: For $\ell \leq \lambda_{min}$ and $\delta < \frac{1}{\Phi_\ell(A)}$

$$\Phi_{\ell+\delta}(A) \leq \frac{1}{1/\Phi_\ell(A) - \delta}$$

is exact when $n=1$.

Idea: at every step, maintain $\ell \leq \mathrm{eigs}(A) \leq u$

$$\Phi_\ell(A) \leq 1, \quad \Phi^u(A) \leq 1$$

Will find s_i to increase, $\ell \leftarrow \ell + 1/3$, $u \leftarrow u + 2$

Update: $\phi^u(A + c\cdot vv^T) = \phi^u(A) + c\,\dfrac{v^T(uI-A)^{-2}v}{1 - cv^T(uI-A)^{-1}v}$

sketch: $\phi^u(A) = \mathrm{Tr}\left[(uI-A)^{-1}\right]$

$\phi^u(A+cvv^T) = \mathrm{Tr}\left[(uI-A-cvv^T)^{-1}\right]$

Sherman-Morrison For nonsingular, vector v, real c

$$(M - cvv^T)^{-1} = M^{-1} + c\,\frac{M^{-1}vv^TM^{-1}}{1-cv^TM^{-1}v}$$

proof substitution.

Cor $\phi^{u+\delta_u}(A+cvv^T) \le \phi^u(A)$ iff

$$\frac{1}{c} \ge \underbrace{v^T\left(\frac{((u+\delta_u)I-A)^{-2}}{\phi^u(A) - \phi^{u+\delta_u}(A)} + ((u+\delta_u)I-A)^{-1}\right)v}_{\triangleq\, U_A}$$

smaller c makes this easier

Is quadratic form in v!

Def $L_A = \dfrac{(A - \ell I)^{-2}}{\phi_{\ell+\delta_L}(A) - \phi_\ell(A)} - (A - (\ell+\delta)I)^{-1}$

Lem $\phi_{\ell+\delta}(A + cvv^T) \le \phi_\ell(A)$

iff $v^T L_A v \ge \frac{1}{c}$. wants c big.

Goal: show $\exists$ i and c s.t.

$$\phi_{\ell+1/3}(A + c v_i v_i^T) \le \phi_\ell(A)$$

and $\phi^{u+2}(A + c v_i v_i^T) \le \phi^u(A)$

Suffices to find i and c so that

$$v_i^T L_A v_i \ge \frac{1}{c} \ge v_i^T U_A v_i$$

First find i s.t. $v_i^T L_A v_i \ge v_i^T U_A v_i$,

and choose c s.t. $\frac{1}{c}$ is between them.

$c > 0$ because $U_A \succcurlyeq 0$

Lem 1. $\sum_i v_i^T U_A v_i \le \frac{1}{\delta_u} + \phi_u(A) = 3/2$

Lem 2. $\sum_i v_i^T L_A v_i \ge \frac{1}{\delta_L} - \frac{1}{1/\phi_\ell(A) - \delta_L} = 3 - \frac{1}{1-1/3} = 3/2$

proof 1 $\sum_i v_i^T U_A v_i = \mathrm{Tr}(U_A)$

$$\mathrm{Tr}\left[\left((u+\delta)I - A\right)^{-1}\right] = \phi^{u+\delta}(A) \le \phi^u(A) \le 1$$

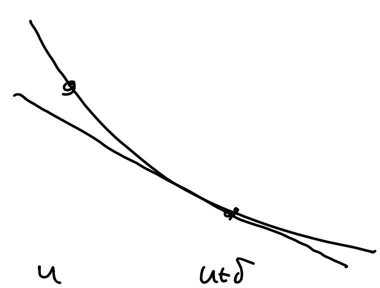
$$\mathrm{Tr}\left[\frac{\left((u+\delta)I - A\right)^{-2}}{\phi^u(A) - \phi^{u+\delta}(A)}\right] \quad (*)$$

$$\frac{\partial}{\partial u}\phi^u(A) = \sum_i \frac{-1}{(u-\lambda_i)^2} = -\mathrm{Tr}(uI-A)^{-2}$$

By convexity

$$\phi^u(A) - \phi^{u+\delta}(A) \ge -\delta \frac{\partial}{\partial u}\phi^{u+\delta}(A) = \delta\,\mathrm{Tr}\left((u+\delta)I - A\right)^{-2}$$

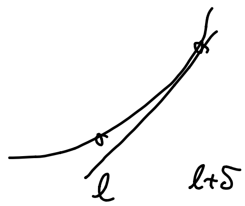
So, $(*) \le \frac{1}{\delta}$



$\mathrm{Tr}(LA)$:

$$\frac{\mathrm{Tr}\left(A-(l+\delta)I\right)^{-2}}{\Phi_{l+\delta}(A)-\Phi_l(A)} \quad (*) \qquad \frac{\partial}{\partial l}\Phi_l(A) = \mathrm{Tr}\left(A-lI\right)^{-2}$$

By convexity



$$\Phi_{l+\delta}(A)-\Phi_l(A) \le \delta \frac{\partial}{\partial l}\Phi_l(A) = \partial\,\mathrm{Tr}(A-lI)^{-2}$$

$$\Rightarrow \quad (*) \ge \frac{1}{\delta}$$

For other term:

$$\mathrm{Tr}\left(A-(l+\delta)I\right)^{-1} \le \frac{1}{1/\Phi_l(A)-\delta}$$