Today: 1. Spectral Theorem as Optimization.

2. The Laplacian

3. Spectral graph drawing

4. The proof

Thm 1 Let M be a symmetric matrix and let x maximize $\frac{x^T M x}{x^T x}$. Then x is an eigenvector of M of eigenvalue $\mu = \frac{x^T M x}{x^T X}$.

proof There is a max, and it is achieved. Suffices to consider $\|x\| = 1$, and the set of unit vectors is closed and compact.

At the max, gradient is 0.

$$\nabla x^T x = 2x \qquad \nabla x^T M x = 2Mx$$

$$\nabla \frac{x^T M x}{x^T x} = \frac{(x^T x)(2Mx) - (x^T M x)(2x)}{(x^T x)^2} = 0$$

$$\Leftrightarrow (x^T x)(Mx) = (x^T M x)\, x$$

$$\Leftrightarrow Mx = \frac{x^T M x}{x^T x}\, x$$

Thm 2 Let M be a symmetric matrix. There exist $\mu_1, \dots, \mu_n$ and orthonormal vectors $\psi_1, \dots, \psi_n$ s.t. $M\psi_i = \mu_i \psi_i$.

Moreover,
$$\psi_i = \arg\max_{\substack{\|x\|=1 \\ x^T\psi_j = 0,\ j<i}} \frac{x^T M x}{x^T x}$$

proof Get μ_1, ψ_1 from Thm 1.

If consider $M + bI$, has same eigvecs as M.

For big b is positive definite $-$ $x^T(M+bI)x > 0, \forall x$.

So, suffices to consider positive definite case.

Assume have $\psi_1 \dots \psi_k$, $\mu_1 \dots \mu_k$. Now construct ψ_{k+1}.

$$M_k = M - \sum_{i=1}^{k} \mu_i \psi_i \psi_i^T$$

Will apply Thm 1 to M_k.

For $x \perp \psi_1, \dots, \psi_k$, $M_k x = Mx$

So
$$\arg\max_{\substack{\|x\|=1 \\ x \perp \psi_1 \dots \psi_k}} \frac{x^T M x}{x^T x} = \arg\max_{\substack{\|x\|=1 \\ x \perp \psi_1, \dots, \psi_k}} \frac{x^T M_k x}{x^T x} \neq \arg\max_{\|x\|=1} \frac{x^T M_k x}{x^T x}$$

Let $y \in \arg\max \frac{x^T M_k x}{x^T x}$ (*)

Will show $y \perp \psi_1 \ldots \psi_k \Rightarrow M_k y = \mu y$ by Thm 1

and $My = M_k y = \mu y$

so, set $\psi_{k+1} = y$, $\mu_{k+1} = \mu$

$$(*) \Rightarrow y \in \underset{x \perp \psi_1 \ldots \psi_k}{\arg\max} \frac{x^T M x}{x^T x}$$

for $j \le k$, $M_k \psi_j = M\psi_j - \sum_{i=1}^{k} \mu_i \psi_i \underbrace{\psi_i^T \psi_j}$,

$= \mu_j \psi_j - \mu_j \psi_j = 0$

Let $\tilde{y} = y - \sum_{i=1}^{k} \psi_i (\psi_i^T y)$ = proj orth to $\psi_1 \ldots \psi_k$

If $\tilde{y} \ne y$ then $\|\tilde{y}\| < \|y\| = 1$ But $\tilde{y}^T M_k \tilde{y} = y^T M_k y$

Now, set $\hat{y} = \frac{\tilde{y}}{\|\tilde{y}\|}$. set $\hat{y}^T M_k \hat{y} > y^T M_k y$,

a contradiction.

Laplacians   Want symmetric matrix L s.t.

$$x^T L x = \sum_{(a,b)\in E} w_{a,b} (x(a) - x(b))^2$$

Consider $G_{a,b}$ - graph with edge (a,b)

$$x^T L_{G_{a,b}} x = (x(a) - x(b))^2 = ((\delta_a - \delta_b)^T x)^2$$

$$= x^T (\delta_a - \delta_b)(\delta_a - \delta_b)^T x$$

$$(\delta_1 - \delta_2)(\delta_1 - \delta_2)^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Is how we set $L = \sum_{(a,b)\in E} w_{a,b} (\delta_a - \delta_b)(\delta_a - \delta_b)^T$

$= D - M$ where D diagonal

$$D(a,a) = \sum_{b:(a,b)\in E} w_{a,b}$$

$$M(a,b) = \begin{cases} w_{a,b} & (a,b)\in E \\ 0 & \text{o.w.} \end{cases}$$

$$(Lx)(a) = d(a)x(a) - \sum_b w_{a,b} x(b) = \sum_{b:(a,b)\in E} w_{a,b} (x(a) - x(b))$$

So, $L\mathbb{1} = 0$. Can see by $M\mathbb{1} = d$ and $D\mathbb{1} = d$.

As $x^T L x \geq 0$, $\forall x$, $\lambda_1 = 0$ is smallest eigval.

Note: order Laplacian eigvals $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$

<u>Thm</u> $\lambda_2 = 0$ iff G is disconnected.

If G is disconnected, can create two orthogonal vectors with eigenvalue zero.

Let $S \subseteq V$ and $T \subseteq V$ be disjoint components.

$$\mathbb{1}_S(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{o.w.} \end{cases}$$

$L\mathbb{1}_S = 0$, as all nbrs of vertices in S are in S.

$L\mathbb{1}_T = 0$. $\mathbb{1}_S^T \mathbb{1}_T = 0$

If G is connected, $\lambda_2 > 0$. Let x be any non-constant vector. Claim is $(a,b) \in E$ s.t. $x(a) \neq x(b)$. Because $\exists\, a, b$ s.t. $x(a) \neq x(b)$, and they are joined by a path.

$\Rightarrow (x(a) - x(b))^2 > 0 \Rightarrow x^T L x > 0$.

Spectral Graph Drawing

In line. Want to choose $x(a) \in \mathbb{R}$ s.t.
neighbors are close.

Consider $\min x^T L x$.

Solution could be zero.

So require $\|x\| = 1$

Solution could be $\frac{1}{\sqrt{n}}\mathbb{1}$. So require $\sum_a x(a) = 0 = 1^T x$.

Now $\psi_2 \in \underset{\substack{\|x\|=1 \\ x^T 1 = 0}}{\arg\min} \; x^T L x$

In 2D, map a to $(x(a), y(a))$

And, $\min \sum_{(a,b) \in E} \left\| \begin{pmatrix} x(a) \\ x(b) \end{pmatrix} - \begin{pmatrix} y(a) \\ y(b) \end{pmatrix} \right\|^2 = x^T L x + y^T L y$

s.t. $\|x\| = 1$

$\|y\| = 1$

$x^T 1 = 0$

$y^T 1 = 0$

$x^T y = 0$

Would guess $x = \psi_2 \quad y = \psi_3$, or any rotation of these

Needs a little work, but is right answer.

In k dimensions, want $x_1 \ldots x_k$ orthonormal s.t.

$$x_i^T \mathbb{1} = 0, \quad \forall i$$

min obj val is $\sum_{i=1}^{k} x_i^T L x_i = \sum_{i=2}^{k+1} \lambda_i$

<u>Thm</u> If $x_1 \ldots x_k$ are orthonormal and orthogonal to $\mathbb{1}$,

then $\sum_{i=1}^{k} x_i^T L x_i \geq \sum_{i=2}^{k+1} \lambda_i$

<u>proof</u>

Choose $x_{k+1}, \ldots, x_n$ to be an orthonormal basis of space orth to $x_1, \ldots, x_k$, so $x_1 \ldots x_n$ is an orthonormal basis.

So, for all j $\sum_i \left(x_i^T \psi_j\right)^2 = 1$ and $\forall i$ $\sum_j \left(\psi_j^T x_i\right)^2 = 1$

$$x_i^T L x_i = \sum_{j=2}^{n} \lambda_j \left(\psi_j^T x\right)^2$$

$$= \lambda_{k+1} + \sum_{j=2}^{n} \left(\lambda_j - \lambda_{k+1}\right) \underbrace{\left(\psi_j^T x_i\right)^2}_{} \quad \text{as} \quad \underbrace{}_{} = 1$$

$$\geq \lambda_{k+1} + \sum_{j=2}^{k+1} \underbrace{\left(\lambda_j - \lambda_{k+1}\right)}_{\text{pos for } j > k+1} \left(\psi_j^T x_i\right)^2$$

$$\sum_{i=1}^{k} x_i^T L x_i \geq k\lambda_{k+1} + \sum_{j=2}^{k+1} \underbrace{\left(\lambda_j - \lambda_{k+1}\right)}_{\text{non-pos}} \underbrace{\sum_{i=1}^{k} \left(\varphi_j^T x_i\right)^2}_{\leq 1}$$

$$\geq k\lambda_{k+1} + \sum_{j=2}^{k+1} \left(\lambda_j - \lambda_{k+1}\right)$$

$$= \sum_{j=2}^{k+1} \lambda_j .$$