Notation: $a,b$ means $(a,b) \in E$
For $S \subseteq V$, $G(S)$ is the induced subgraph on $S$.
It has vertex set $S$ and edges $\{ (a,b) \in E : a \in S \text{ and } b \in S \}$

Adjacency matrices, eigenvalue interlacing, PF theory.
$M(a,b) = \begin{cases} \omega_{ab} & \text{if } (a,b) \in E \\ 0 & \text{o.w.} \end{cases}$

Denote eigvals $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$
Reason: for regular, $L = dI - M$
so $\lambda_i = d - \mu_i$, $\lambda_0 \leq \lambda_2 \leq \ldots \leq \lambda_n$

For irregular, $\mu_1$ and $\mu_n$ eigen are more interesting.

Then $d_{\text{ave}} \leq \mu_1 \leq d_{\text{max}}$.

Proof: By CF

$$\mu_1 = \max_{x} \frac{x^T M x}{x^T x} \geq \frac{1^T M 1}{1^T 1} = \frac{1^T d}{n} = \frac{\sum_{a \in V} d(a)}{n} = \text{dave}$$

Let $\Psi_1$ be eigvec at $\mu_1$.
Let $a = \arg \max \Psi_1(a)$. Then

$$\mu_1 \cdot \Psi_1(a) = \sum_{b \neq a} \omega_{ab} |\Psi_1(b)| \leq \sum_{b \neq a} \omega_{ab} \Psi_1(a) \leq d(a) \cdot \Psi_1(a) \leq d_{\text{max}} \cdot \Psi_1(a)$$

$\mu_1$ also says something about subgraphs.
Cayley's Interlacing Theorem

Let \( A \) be a matrix and let \( B \) be a principal submatrix obtained by deleting a row and column.

\[
eig(A) = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n, \quad \eig(B) = \beta_1 \geq \beta_2 \geq \ldots \geq \beta_{n-1}
\]

Then \( \lambda_k \geq \beta_k \geq \lambda_2 \geq \beta_2 \geq \ldots \geq \lambda_{k-1} \geq \beta_{k-1} \geq \lambda_n \).

Proof: Assume remove first row/col to get \( B \).

First, \( \lambda_k \geq \beta_k \).

\[
CE = \lambda_k = \max_{S \subset \text{dim} \leq k} \min_{x \in \mathbb{R}^n} \frac{x^T A x}{x^T x}
\]

\[
\beta_k = \max_{S \subset \text{dim} \leq k-1} \min_{x \in \mathbb{R}^n} \frac{x^T B x}{x^T x} = \max_{S \subset \text{dim} \leq k-1} \min_{x \in \mathbb{R}^n} \frac{(0)^T B x}{(0)^T x}
\]

\[
\leq \lambda_k
\]

\( \lambda_{k+1} \leq \beta_k \) follows by changing \( A \) and \( B \) to \(-A\) and \(-B\).

If remove a vertex, obtain submatrix as odd matrix. \( \mu_1 \) can only go down, while case can go up or down.

Prop 2: For every \( S \in V \), \( \text{dave}(G(S)) = \mu_1 \).

Proof: Cayley \( \Rightarrow \) max eig of \( M(S) \) is \( \mu_1 \), and can apply Thm 1.
Proof: Graph coloring.
A $k$-coloring of $G$ is $C : \{1..k\} \rightarrow \mathcal{V}$ s.t.
\[ (a, b) \in E \Rightarrow C(a) \neq C(b) \]
\[ x(G) = \min \{ k : \text{G is k-colorable} \} \]
Easy: $x(G) = d_{max} + 1$, by greedy.

Wilt: $x(G) \leq \lceil \mu \rceil + 1$.

Note $\mu$ can be much less than $d_{max}$.
\[ \mu_1 = \frac{\alpha}{2}, \quad d_{max} = 2 \]
with $n + 1$ vertices, $\mu_1 = \sqrt{n}$, $d_{max} = n$.

Proof: by induction on $|V|$. Base case ($|V| = 1$).
\[ x(G) = 1, \quad \mu_1 = 0 \]

Induction: We know $d_{min} \leq d_{max} \leq \mu_1$. So
$G$ has a vertex, $\alpha$, with $d(\alpha) \leq \mu_1$. Let $S = V - \{\alpha\}$.

Conclude: $\mu_1(G(S)) \leq \mu_1(G)$, so induct hyp $\Rightarrow$
can color $G(S)$ with $\lceil \mu_1 + 1 \rceil$ colors.
Remains to pick a color for $\alpha$. It has $\leq \lfloor \mu_1 \rfloor$ neighbors,
so a color is available.
PF Theory

The eigenvector of $\mu_1$.

Then let $G$ be a connected weighted graph.

a. $\mu_1$ has a strictly positive eigenvector.

b. $\mu_1 = |\mu_1|$

c. $\mu_1 > \mu_2$

Lemma 3. Let $G$ be connected and let $\psi$ be a non-negative eigenvector of $\mu$. Then $\psi$ is strictly positive.

Proof. Assume, true, $\exists a : \psi(a) = 0$.

As $G$ connected $\Rightarrow \exists (a,b)$ such that $\psi(a) = 0 < \psi(b)$.

Let $\mu$ be the equal. Then

$$\mu \cdot \psi(a) = \sum_{z \sim a} M(a,z) \psi(z) \geq M(a,b) \psi(b) > 0, \neq\quad\text{#}$$

Proof of $\phi_i$.

$\psi_i$ is eigenvector of $\mu_i$, $x(a) = |\psi_i(a)| \forall a$.

Will show $x$ is eigenvector of $\mu_i$. Lemma 3 $\Rightarrow$ strictly positive

$$\mu_i = \phi_i^T M \phi_i = \sum_{(a,b) \in E} M(a,b) \phi_i(a) \phi_i(b)$$

$$\leq \sum_{a \in V} M(a,b) |\phi_i(a)| |\phi_i(b)| = x^T M x \leq \mu_i$$

Now, use maximum $\Rightarrow$ eigenvector.
Proof of b. Let \( \psi_a = \text{esvec of } \mu_a \), \( \gamma(\alpha) = |\psi_a(\alpha)| \), \( \theta_\alpha \).

\[
\begin{align*}
|\mu_a| = |\psi_a^\top M \psi_a| = \sum_{a,b} M(a,b) \gamma(\alpha) \gamma(\beta) \leq \gamma^\top M \gamma \leq \mu_1
\end{align*}
\]

proof of c. \( \mu_2 < \mu_1 \).

Let \( \psi_2 \) be esvec of \( \mu_2 \). As \( \Re \psi_a = \psi_1 \), \( \psi_2 \) has pos and neg entries. Let \( \gamma(a) = |\psi_2(a)| \), \( \theta_\alpha \).

\[
\mu_2 = \psi_2^\top M \psi_2 \leq \gamma^\top M \gamma \leq \mu_1
\]

If \( \mu_2 = \mu_1 \), \( \gamma \) is nonneg esvec of \( \mu_1 \), and thus strictly positive \( \Rightarrow \psi_2 \) is never zero

\[
\Rightarrow \exists \{a,b\} \text{ s.t. } \gamma(\alpha) \geq 0 < \gamma(\beta).,
\]

\[
\Rightarrow \psi_2^\top M \psi_2 < \gamma^\top M \gamma, \text{ as } M(a,b) \gamma_\alpha(\gamma_\beta) \gamma(\beta) < 0 < M(a,b) \gamma_\alpha(\gamma_\beta) \gamma(\beta).
\]

Contradiction.

What if \( \mu_2 = -\mu_1 \)?

Thus If \( G \) connected, \( \mu_2 = -\mu_1 \) iff \( G \) is bipartite.

If \( \mu_2 = -\mu_1 \), \( \gamma \) is tight

\[
\Rightarrow \gamma \text{ an esvec of } \mu_1, \text{ strictly positive },
\]

\[
\Rightarrow \psi_2 \text{ is never zero, and } \exists \{a,b\} \text{ s.t. } \gamma_\alpha(\gamma_\beta) < 0 < \gamma_\alpha(\gamma_\beta).
\]
\begin{align*}
\sum_{a \sim b} M(a, b) \psi(a) \psi(b) &= \sum_{a \sim b} M(a, b) |\psi(a)|^2 |\psi(b)|^2 \\
\Rightarrow\text{ all terms in } & \text{ have same sign. Edge } (a, b) \Rightarrow \text{ negative.} \\
\Rightarrow\text{ so, } & \psi(a) \psi(b) < 0 \text{ all } (a, b) \in E \\
\Rightarrow\text{ signs give bipartition.} \\
\text{Bipartite } & \iff \mu \text{ is real } \iff -\mu \text{ is real.} \\
\text{can order vertices so } M &= \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \\
\text{let } M(x_0) &= M(x_1), \text{ so } \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} x_0 = \begin{pmatrix} Bx_1 \\ B^T x_0 \end{pmatrix} = \begin{pmatrix} \mu x_0 \\ \mu x_1 \end{pmatrix} \\
\text{then } M(-x_0) &= \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} (-x_0) = \begin{pmatrix} -Bx_1 \\ B^T x_0 \end{pmatrix} = M(-x_1) = -M(x_1) \\
\text{so, } & -\mu \text{ is an eigenvalue.}
\end{align*}