Comparing Graphs. Lower bounds on $\lambda_2$.

$A \succ 0$ if $A$ is positive semidefinite:
symmetric with no negative eigenvalues.

$\Rightarrow \forall x, x^T Ax \geq 0, \forall x.$

$A \succeq B \iff A-B \succ 0.$
Is Loewner partial order.

$A \succeq B$ and $B \succeq C \Rightarrow A \succeq C.$

And for all symmetric $C$

$A \succeq B \Rightarrow A+C \succeq B+C$

Overload: define for graphs.

$G \succeq H \iff LG \succeq LH$

Recall $LG = \sum_{a,b} w_{a,b} (x(a) - x(b))^2$

So, if $H$ is a subgraph of $G$, $H \subseteq G$

Same if decrease weights to get $H$.

Often consider $G \succeq cH$ where $c > 0,$

$cH$ is $H,$ but weight edge sets by $c$

$\lambda_{cH} = c\lambda_H$
Theorem: If $G \geq cH$, then $\chi_e(G) \geq c \chi_e(H)$ for all $k$.

**Proof** by CF:

$$\chi_e(G) = \min_{\dim(S) = k} \max_{x \in S} \frac{\chi^T Ce x}{x^T x} \geq \min_{\dim(S) = k} \max_{x \in S} \frac{c \chi^T H e x}{x^T x}$$

$$= C \chi_e(H)$$

Data inequality: \( (n-1) P_n \geq G_{1,n} \)

\( P_n \) is path from 1 to n. \( G_{1,n} \) just has edge \( C_{1,n} \).

**Proof** \( y \in \mathbb{R}^n \), need to show:

$$\sum_{a=1}^{n-1} (x(a+1) - x(a))^2 \geq (x(n) - x(1))^2$$

Set \( \Delta(a) = x(a+1) - x(a) \), so \( x(n) - x(1) = \sum_{a=1}^{n-1} \Delta(a) \).

$$\sum_{a=1}^{n-1} \Delta(a)^2 = \left( \sum_{a=1}^{n-1} \Delta(a) \right)^2 \geq \left( \sum_{a=1}^{n-1} A(a) \right)^2$$

Thus Cauchy Schwartz:

$$\left( \sum_{a=1}^{n-1} \Delta(a) \right)^2 = (1^{T_n} \Delta)^2 \leq \|1_{n-1}\| \|\Delta\| = (n-1) \sum_{a=1}^{n-1} \Delta(a)^2$$
Now, let's see how to use this to prove a lower bound on $\lambda_2(P_n)$.

Last class saw $\lambda_2(P_n) \approx \frac{\pi^2}{n^2}$ and $\lambda_2(P_n) \leq \frac{12}{n(n+1)}$.

To lower bound will prove

$P_n \geq cK_n$ and recall $\lambda_2(K_n) = n$,

so implies $\lambda_2(P_n) \geq cn$.

Write $K_n = \sum_{a \neq b} G_{a,b}$

For $a \neq b$, write $P_{a,b}$ for subgraph of $P_n$ from $a$ to $b$

$G_{a,b} \leq (b-a)P_{a,b} \leq (b-a)P_n$

$\Rightarrow K_n = \sum_{a \neq b} G_{a,b} \leq \sum_{a \neq b} (b-a)P_n = \frac{n(n+1)(n-4)}{6} P_n$

So,

$P_n \geq \frac{6}{n(n+1)(n-1)}$

$\Rightarrow \lambda_2(P_n) \geq n \cdot \frac{6}{(n+1)(n-1)n} = \frac{6}{(n+1)(n-1)}$
Complete binary tree. \( T_n \) \( n = 2^{d+1} - 1 \)

edges \( a \rightarrow 2q \)
\( a \rightarrow 2a + 1 \)

depth, 3

Upper bound on \( \lambda_2 (T_n) \) by

\[
\frac{x^T L x}{x^T x} = \frac{2}{n-1} \geq \lambda_2
\]

Lower bound:
For each \( a \neq b \), let \( T_{iba} \) be the path in \( T_n \) from \( a \) to \( b \).
Is unique because it is a tree, and has length \( \leq 2d \)
\( \leq 2 \lg n \)

\[
K_n = \sum_{a \neq b} G_{a \rightarrow b} \leq \sum_{a \neq b} (2d) T_{iba} \leq \sum_{a \neq b} (2d) T_n
\]

\[
\leq \binom{n}{2} (2 \lg n) T_n = n(n-1) \lg n T_n
\]

\[
\Rightarrow \lambda_2 (T_n) \geq \frac{1}{n(n-1) \lg n} \text{det} K_n = \frac{1}{(n-1) \lg n}
\]

Differs from test vector bound by \( \lg n \)
Sometimes use many paths.
Sometimes weight them.

Let $P$ be the weighted path with edges $(a, a+i)$ at at $u_a$.

Then $G_{in} = \left( \sum_{a=1}^{k} \frac{1}{u_a} \right) P$

**Proof** For $x \in \mathbb{R}^n$, let $\Delta_i(a) = x(a+i) - x(a)$.

Set $\Delta(a) = \Delta_i(a) \omega_a$

$w^{-1/2}$ be vector s.t. $w^{1/2} \Delta(a) = \omega(a)^{1/2}$

Then $\sum_a \Delta(a) = 2T \omega^{-1/2}$

so $x^T G_{in} x = \left( \sum_a \Delta(a) \right)^2 = \left( 2T \omega^{-1/2} \right)^2$

\[ = \left( \|w^{-1/2}\| \|x\| \right)^2 \]

\[ = \left( \sum_{a=1}^{k} \frac{1}{u_a} \right) \left( \sum_a \omega_a \Delta(a)^2 \right) \]

\[ = \left( \sum_{a=1}^{k} \frac{1}{u_a} \right) x^T L_p x \]

**Remark** Is resistance of resistor in series, with $r(a) = \frac{1}{\omega(a)}$. 
Approximations of graphs.

\[ \frac{1}{2} H \leq G \leq dH \]

Next lecture: a random graph, every edge prob \( p \)

is a good \( (1+\varepsilon) \)-approximation of \( K_n \)

Expanders: For every \( \varepsilon > 0 \), a constant \( d \)

s.t. there is a \( d \)-regular \( (1+\varepsilon) \)-approx of \( K_n \)

Ramanujan: \( \varepsilon = \frac{2}{\sqrt{d}} \)

Sparsifiers: For every \( \varepsilon > 0 \), a constant \( d \)

s.t. every \( G \) on \( n \) vertices has a \( (1+\varepsilon) \)-approx

with \( \frac{dn}{2} \) (weighted) edges.