Comparing Graphs. Lower bounds on λ_2.

$A \succcurlyeq 0$ if A is positive semidefinite:
symmetric with no negative eigenvalues.

$\iff x^T A x \geq 0, \ \forall x.$

$A \succcurlyeq B$ if $A - B \succcurlyeq 0.$
Is Loewner partial order.

$A \succcurlyeq B$ and $B \succcurlyeq C \Rightarrow A \succcurlyeq C.$

And for all symmetric C
$A \succcurlyeq B \Rightarrow A + C \succcurlyeq B + C$

Overload: define for graphs.
$G \succcurlyeq H$ iff $L_G \succcurlyeq L_H$

Recall $L_G = \sum_{a \sim b} w_{a,b} \left(x(a) - x(b)\right)^2$

So, if H is a subgraph of G, $H \preccurlyeq G$

Same if decrease weights to ~~get~~ H.

Often consider $G \succcurlyeq cH$ where $c > 0$,
cH is H, but mult edge wts by c

$L_{cH} = c L_H$

Theorem If $G \succeq cH$ then

$\lambda_k(G) \geq c\lambda_k(H)$ for all k

proof by CF

$$\lambda_k(G) = \min_{\dim(S)=k} \max_{x \in S} \frac{x^T L_G x}{x^T x}$$

$$\geq \min_{\dim(S)=k} \max_{x \in S} c\frac{x^T L_H x}{x^T x}$$

$$= c\,\lambda_k(H)$$

Path inequality: $(n-1)P_n \succeq G_{1,n}$

P_n is path from 1 to n. $G_{1,n}$ just has edge $(1,n)$

proof $\forall x \in \mathbb{R}^n$, need to show

$$(n-1)\sum_{a=1}^{n-1}(x(a+1)-x(a))^2 \geq (x(n)-x(1))^2$$

Set $\Delta(a) = x(a+1)-x(a)$, so $x(n)-x(1) = \sum_{a=1}^{n-1}\Delta(a)$.

NTS $(n-1)\sum_{a=1}^{n-1}\Delta(a)^2 \geq \left(\sum_{a=1}^{n-1}\Delta(a)\right)^2$. Is Cauchy Schwartz.

$$\left(\sum_{a=1}^{n-1}\Delta(a)\right)^2 = \left(1_{n-1}^T\Delta\right)^2 \leq \|1_{n-1}\|^2\|\Delta\|^2 = (n-1)\sum_{a=1}^{n-1}\Delta(a)^2$$

Now, lets see how to use this to prove a lower bound on $\lambda_2(P_n)$.

Last class saw $\lambda_2(P_n) \approx \frac{\pi^2}{n^2}$ and $\lambda_2(P_n) \leq \frac{12}{n(n+1)}$

To lower bound will prove

$P_n \succcurlyeq c K_n$, and recall $\lambda_2(K_n) = n$,

so implies $\lambda_2(P_n) \geq cn$

Write $K_n = \sum_{a<b} G_{a,b}$

For $a<b$, write $P_{a,b}$ for subgraph of P_n from a to b

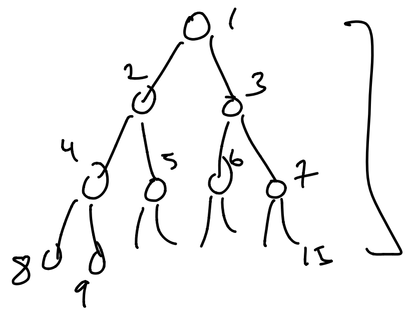
$$G_{a,b} \preccurlyeq (b-a) P_{a,b} \preccurlyeq (b-a) P_n$$

$$\Rightarrow K_n = \sum_{a<b} G_{a,b} \preccurlyeq \sum_{a<b} (b-a) P_n = \frac{n(n+1)(n-1)}{6} P_n$$

so, $P_n \succcurlyeq \frac{6}{n(n+1)(n-1)}$

$$\Rightarrow \lambda_2(P_n) \geq n \cdot \frac{6}{(n+1)(n-1)n} = \frac{6}{(n+1)(n-1)}$$

Complete binary tree. T_n $\qquad n = 2^{d+1}-1$

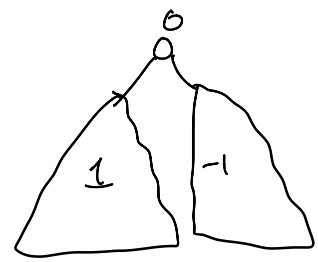


edges $a \to 2a$

$a \to 2a+1$

depth 3

Upper bound on $\lambda_2(T_n)$ by



$$\frac{x^T L x}{x^T x} = \frac{2}{n-1} \geq \lambda_2$$

Lower bound:

For each $a<b$, let $T_n^{a,b}$ be the path in T_n from a to b.

Is unique because is a tree, and has length $\leq 2d$

$\leq 2\lg_2 n$

$$K_n = \sum_{a<b} G_{a,b} \leq \sum_{a<b} (2d) T_n^{a,b} \leq \sum_{a<b} (2d) T_n$$

$$\leq \binom{n}{2} (2\lg_2 n) T_n = n(n-1)\lg_2 n \, T_n$$

$$\Rightarrow \lambda_2(T_n) \geq \frac{1}{n(n-1)\lg_2 n} \lambda_2(K_n) = \frac{1}{(n-1)\lg_2 n}$$

Differs from test vector bound by $\lg_2 n$

Sometimes use many paths.

Sometimes weight them.

Let P be the weighted path with edges $(a, a+1)$ of wt w_a.

Thm $G_{1,n} \preccurlyeq \left(\sum_{a=1}^{n-1} \frac{1}{w_a} \right) P$

proof For $x \in \mathbb{R}^n$, let $\Delta(a) = x(a+1) - x(a)$.

Set $\gamma(a) = \Delta(a) \sqrt{w_a}$

$w^{-1/2}$ be vector s.t. $w^{-1/2}(a) = w(a)^{-1/2}$

Then $\sum_a \Delta(a) = \gamma^T w^{-1/2}$

so
$$
\begin{aligned}
x^T G_{1,n} x &= \left(\sum_a \Delta(a) \right)^2 = \left(\gamma^T w^{-1/2} \right)^2 \\
&\leq \| w^{-1/2} \|^2 \, \| \gamma \|^2 \\
&= \left(\sum_{a=1}^{n-1} \frac{1}{w(a)} \right) \cdot \sum_a w_a \Delta(a)^2 \\
&= \left(\sum_{a=1}^{n-1} \frac{1}{w(a)} \right) \cdot x^T L_P x
\end{aligned}
$$

$\boxed{\text{Remark}}$ Is resistance of resistors in series, with $r(a) = \frac{1}{w(a)}$.

Approximations of graphs.

H is a c-approximation of G if

$$\frac{1}{c}H \preceq G \preceq cH$$

Next lecture: a random graph, every edge prob ½
is a good $(1+\varepsilon)$, small ε - approximation of K_n

Expanders: For every $\varepsilon > 0$, a constant d
s.t. there is a d-regular $(1+\varepsilon)$-approx of K_n

Ramanujan: $\varepsilon \approx \frac{2}{\sqrt{d}}$

Sparsifiers: For every $\varepsilon > 0$, is a constant d
s.t. every G on n vertices has a $(1+\varepsilon)$-approx
with $\frac{dn}{2}$ (weighted) edges.