Erdős–Renyi. Each edge pub p

\[ \mu_i = p_n \quad |\mu_i| = \left(1+\varepsilon\right)2^{-\frac{1}{2}n+O(1)} \quad i=2, \ldots \]

\[ R(i,j) = \begin{cases} 1-p & \text{pub p} \\ -p & \text{pub i-p} \end{cases} \quad E R(i,j) = 0 \]

\[ M = PJ - pI + pR \quad J = \text{all Is matrix} \]

\[ EM = p(J-I) \]

let \( \lambda_i \ldots \lambda_n \) be eigenval of \( R \).

\[ \sum \lambda_i^k = \text{Tr}(R^k) \]

Core about \( \|R\| = \max_i |\lambda_i| \)

\[ \|R\|_F \leq \text{Tr}(R^2) \]

\[ \|R\| \leq \left(\text{Tr}(R^2)\right)^{1/2} \]

We will prove if \( np(1-p) \geq 2 - k^8 \)

\[ E \text{Tr}(R^k) \leq 2n \left(2^{-\frac{1}{2}n+O(1)}\right)^k \leq \lambda^k \quad \lambda = (2n)^{1/2} 2^{-k/4} \]

So, for \( \varepsilon > 0 \)

\[ \Pr \left[ \|R\| \geq (1+\varepsilon)\lambda \right] \leq \Pr \left[ \text{Tr}(R^k) \geq (1+\varepsilon)^k \lambda^k \right] \]

\[ \leq \Pr \left[ \text{Tr}(R^k) \geq (1+\varepsilon)^k E \text{Tr}(R^k) \right] \]
\[ \varepsilon \leq \frac{1}{((\varepsilon / \ell)\ell)} e^{-\varepsilon \ell} \]

so small if \( \varepsilon < \frac{1}{\ell} \)

\[ (2\pi)^{d/2} = \exp \left( \frac{\ln(2\pi)}{2} \right) \times \left( 1 + \frac{\ln(2\pi)}{e} \right) \] so close to 1

\[ R^l(a_0, a_0) = \sum_{a_1} R(a_0, a_1) R^{l-1}(a_1, a_0) \]
\[ = \sum_{a_1} R(a_0, a_1) \prod_{i=1}^{l-1} R(a_i, a_{i+1}) \]
\[ E R^l(a_0, a_0) = \sum_{a_1, \ldots, a_{l-1}} E R(a_0, a_1) \prod_{i=1}^{l-1} R(a_i, a_{i+1}) \]

For indep \( X, Y \) \( E[XY] = EX \cdot EY \)

and \( E R(a, b) = 0 \).
So, only nonzero when each term appears at least twice

Let \( \{ b_j, c_j \} \) be the distinct pairs in \( \{ a_0, a_2, \ldots, a_{l-1} \} \)
and let \( \{ b_j, c_j \} \) appear \( a_j \) times.

Then \[ \sum_{\{ \}} = \prod_{i=0}^{l} E R(b_j, c_j) \]

\[ E R(b_0, c_0) = P(c_0) \left( (\ell - 1)^{d_{b_0} - 1} - (\ell - 1)^{d_{c_0} - 1} \right) \]

\[ = P(c_0) d_{2} \]
\[ E R(a_0, a_{n-1}) = \text{sum over seq } a_0, a_{n-1} \]
\[ \text{such that each pair occurs at least two times} \]
\[ \frac{1}{(p(1-p))} \text{ distinct pairs.} \]

Say \( a_0, a_1, \ldots, a_n \) is a closed walk of length \( l \) if \( a_0 = a_n \) and it is significant if each pair \( \{a_i, a_{i+1}\} \) occurs at least twice.

\[ W_{n, l, k} = \# \text{ sig closed walks length } l \text{ with } \]
\[ k \text{ distinct elements among } a_0, a_1, \ldots, a_{n-1} \]

\[ \mathbb{E} \text{Tr}(R^k) \leq \sum_{k=1}^{l/2} W_{n, l, k} (p(1-p))^k \]

\[ \# \text{ distinct pairs } \leq \# \text{ distinct elements} \]

Will prove \( W_{n, l, k} \leq \frac{n^{k+1} 2^k l^{n(l-2k)}}{k!} \)

**Theorem:** If \( l \) even and \( np(1-p) = 2l^5 \), then

\[ \mathbb{E} \text{Tr}(R^k) \leq 2n \left( 2^5 np(1-p)^{-1} \right)^k \]

**Proof:** Let \( t_k = n^{k+1} 2^k l^{n(l-2k)} (p(1-p))^k \)

Then \( t_k \geq 2t_{k-1} \) for \( np(1-p) = 2l^5 \)

So \[ \sum_{k=1}^{l/2} W_{n, l, k} (p(1-p))^k \leq \sum_{k=1}^{l/2} t_k \leq 2t_{l/2} = \left( 2n \right)^{l/2} n^{l(l-2l/2)} (p(1-p))^{l/2} \]

\[ = 2n \left( 2^{5} np(1-p)^{-1} \right)^{l/2} \]
Bound on $W_n,k = a_0,a_1,...,a_n,a_0$

$S = \{i : a_i = a_j \text{ for } j < i\}$ "appears first" (k)

For each $i \in S$, record $a_i - \sigma(i) = a_i$

And, record $a_0$

For $i \notin S$, record $j \in S$ s.t. $a_i = a_j$ (k+1)

or $0$. $\zeta : \{0,1\}^n \rightarrow S$

So, $W_n,k,t = \binom{l-1}{k} n^{k-1} \binom{k+1}{l-1-k} \leq n \cdot 2^l \binom{k+1}{l-1-k}$

Example

$k=4$  $S = \{1,2,3,6\}$

$\sigma(1) = d$  $\zeta(1) = 0$

$\sigma(2) = c$  $\zeta(5) = 1$

$\sigma(3) = a$  $\zeta(7) = 1$

$\sigma(6) = e$  $\zeta(8) = 2$

$\zeta(9) = 3$

Step 0 1 2 3 4 5 6 7 8 9 10

Vertex a b d c a b e b d c b

$W_n,k,t \leq n \cdot 2^l \binom{k+1}{l-1-k}$  gets us to $\frac{\sum p(l-1-k)}{ln}$

Can improve for big $k$. 
When \( k = \frac{1}{2} \), do not need \( \tau \).

no choice in \( a_i \) for \( i \notin S \).

Call pair \( (a_{i-1}, a_i) \) to \( i \notin S \) a tree edge.

Because they form a tree. Are \( k = \frac{1}{2} \) \( i \notin S \).

each has an edge. So, \( k \) edges, \( k+1 \) vertices (with \( a_0 \))

and connected \( (\text{by induction}) \)

\[
\begin{array}{c}
1 & 5 \\
2 & 6 \\
\end{array}
\]

Must use each tree edge twice, and no others

Look at tree edges used exactly once through step \( i-1 \).

If \( i \notin S \), will show is exactly one touching \( a_{i-1} \), so must follow it.

Because only use tree edges, and \( i \notin S \) so must be used before

Why only one?

Graph \( G_i = \) Tree edges used exactly once in \( a_0 \ldots a_i \)

vertices attached to them. and \( a_i \)

\( v_i = \# \) vertices in \( G_i \), \( e_i = \# \) edges

\( v_0 = 1 \quad e_0 = 0 \quad v_e = 1 \quad e_e = 0 \)
For $i < s$, $v_i = v_{i-1}$, $e_i = e_{i-1} + 1$.

If $S$, if \( \deg_{G_{i-1}}(a_{i-1}) = 1 \),

- remove \((a_{i-1}, a_i)\) and \(a_{i-1}\), so
  - \(v_i = v_{i-1}\), \(e_i = e_{i-1} - 1\).

If \( \deg_{G_{i-1}}(a_{i-1}) = 2 \),

- remove \((a_{i-1}, a_i)\) but not \(a_{i-1}\).
  - \(v_i = v_{i-1}\), \(e_i = e_{i-1} - 1\).

Contradiction.

So,

\[
W_{k+1} \leq 2^{\frac{k+1}{4}} \cdot n^2
\]

To show \(W_{k+1} \leq 2^{\frac{k+1}{4}} \cdot n^4 (k-2k-1)\).

\(T = \{ i : \{a_{i-1}, a_i\} \in E_{i-1}, \deg_{G_{i-1}}(a_i) = 2 \}\) is the tree.

Step i ambiguous if \(\{a_{i-1}, a_i\} \in E_i\) but \(\deg_{G_{i-1}}(a_i) > 1\),

- \(e_i = e_{i-1} - 1\), \(v_i = v_{i-1}\).

Step i extra if \(\{a_{i-1}, a_i\}\) not tree edge

- or already used \(\geq 2\) times.

\(e_i = e_{i-1}\), \(v_i = v_{i-1} - 1\).

Can go down if \(\deg_{G_{i-1}}(a_{i-1}) = 0\).

For ambiguous or extra, use \(T\) to record \(j\)'s s.t. \(a_i = a_j\).
# ambiguous ≤ # extra

# extra ≤ l - 2k

⇒ \text{spec } S, T_1, S', T, \Theta

\begin{align*}
&\left(\binom{l-1}{k}, \binom{l-1-k}{2^k}, n^k, \binom{k+1}{2^k}\right) \\
&\leq 2^{l-1} (l-1)^{2^k} (k+1)^{2^k} n^{k+1} \\
&\leq 2^{l-1} n^{k+1} l^{4^k} \\
&\leq 2^{l-1} n^{k+1} l^{(l-2)^k}
\end{align*}