**Spectral Graph Theory** Lecture 5

# The other eigenvectors of the Laplacian

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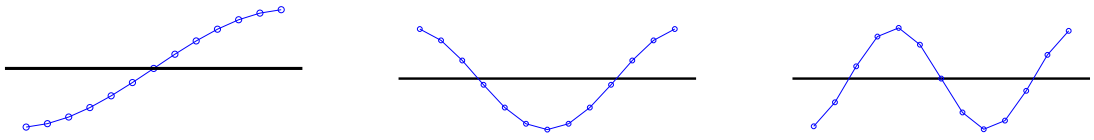
## 5.1 Overview

We are now going to begin our study of the other eigenvalues and eigenvectors of the Laplacian. I will begin the lecture by showing how much of the theory we established can be preserved. We will then determine the eigenvalues of the hypercube, and begin to see why $\lambda_2$ is so important.

## 5.2 Remember the path

Recall that we showed that the $k$th eigenvector of a path graph crosses the origin at most $k-1$ times. For example, here are the first three non-constant eigenvectors of the path graph on 13 vertices, with a line drawn at the origin. Today, we will prove a result of Fiedler [Fie75] which says



that for every $G$ the graph induced on the vertices that are non-negative in the $k$th eigenvector has at most $k-1$ connected components. In particular, it says that the non-negative vertices in $\boldsymbol{v}_2$ are connected.

First, we need to recall a little linear algebra.

## 5.3 The Perron-Frobenius Theorem for Laplacians

In Lecture 3, we proved the Perron-Frobenius Theorem for non-negative matrices. I wish to quickly observe that this theory may also be applied to Laplacian matrices, to principal sub-matrices of Laplacian matrices, and to any matrix with non-positive off-diagonal entries. The difference is that it then involves the eigenvector of the smallest eigenvalue, rather than the largest eigenvalue.

**Corollary 5.3.1.** *Let $M$ be a matrix with non-positive off-diagonal entries, such that the graph of the non-zero off-diagonally entries is connected. Let $\lambda_1$ be the smallest eigenvalue of $M$ and let $\boldsymbol{v}_1$ be the corresponding eigenvector. Then $\boldsymbol{v}_1$ may be taken to be strictly positive, and $\lambda_1$ has multiplicity* 1.

*Proof.* Consider the matrix $A = \sigma I - M$, for some large $\sigma$. For $\sigma$ sufficiently large, this matrix will be non-negative, and the graph of its non-zero entries is connected. So, we may apply the Perron-Frobenius theory to $A$ to conclude that its largest eigenvalue $\alpha_1$ has multiplicity 1, and the corresponding eigenvector $\boldsymbol{v}_1$ may be assumed to be strictly positive. We then have $\lambda_1 = \sigma - \alpha_1$, and $\boldsymbol{v}_1$ is an eigenvector of $\lambda_1$. □

## 5.4 Eigenvalue Interlacing

We will often use the following elementary consequence of the Courant-Fischer Theorem. I recommend deriving it for yourself.

**Theorem 5.4.1** (Eigenvalue Interlacing)**.** *Let $A$ be an $n$-by-$n$ symmetric matrix and let $B$ be a principal submatrix of $A$ of dimension $n-1$ (that is, $B$ is obtained by deleting the same row and column from $A$). Then,*

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1} \geq \alpha_n,$$

*where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1}$ are the eigenvalues of $A$ and $B$, respectively.*

**Corollary 5.4.2** (Eigenvalue Interlacing)**.** *Let $A$ be an $n$-by-$n$ symmetric matrix and let $B$ be a principal submatrix of $A$ of dimension $n-k$ (that is, $B$ is obtained by deleting the same set of $k$ rows and columns from $A$). Then,*

$$\alpha_i \geq \beta_i \geq \alpha_{i+k},$$

*for all $1 \leq i \leq n-k$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-k}$ are the eigenvalues of $A$ and $B$, respectively.*

## 5.5 Fiedler's Nodal Domain Theorem

Given a graph $G = (V, E)$ and a subset of vertices, $W \subseteq V$, recall that the *graph induced by $G$ on $W$* is the graph with vertex set $W$ and edge set

$$\{(i,j) \in E, i \in W \text{ and } j \in W\}.$$

This graph is sometimes denoted $G(W)$.

**Theorem 5.5.1** ([Fie75])**.** *Let $G = (V, E, w)$ be a weighted connected graph, and let $L_G$ be its Laplacian matrix. Let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $L_G$ and let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be the corresponding eigenvectors. For any $k \geq 2$, let*

$$W_k = \{i \in V : \boldsymbol{v}_k(i) \geq 0\}.$$

*Then, the graph induced by $G$ on $W_k$ has at most $k-1$ connected components.*

*Proof.* To see that $W_k$ is non-empty, recall that $\boldsymbol{v}_1 = \mathbf{1}$ and that $\boldsymbol{v}_k$ is orthogonal $\boldsymbol{v}_1$. So, $\boldsymbol{v}_k$ must have both positive and negative entries.

Assume that $G(W_k)$ has $t$ connected components. After re-ordering the vertices so that the vertices in one connected component of $G(W_k)$ appear first, and so on, we may assume that $L_G$ and $\boldsymbol{v}_k$ have the forms

$$L_G = \begin{bmatrix} B_1 & \mathbf{0} & \mathbf{0} & \cdots & C_1 \\ \mathbf{0} & B_2 & \mathbf{0} & \cdots & C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & B_t & C_t \\ C_1^T & C_2^T & \cdots & C_t^T & D \end{bmatrix} \quad \boldsymbol{v}_k = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_t \\ \boldsymbol{y} \end{pmatrix},$$

and

$$\begin{bmatrix} B_1 & \mathbf{0} & \mathbf{0} & \cdots & C_1 \\ \mathbf{0} & B_2 & \mathbf{0} & \cdots & C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & B_t & C_t \\ C_1^T & C_2^T & \cdots & C_t^T & D \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_t \\ \boldsymbol{y} \end{pmatrix} = \lambda_k \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_t \\ \boldsymbol{y} \end{pmatrix}.$$

The first $t$ sets of rows and columns correspond to the $t$ connected components. So, $\boldsymbol{x}_i \geq 0$ for $1 \leq i \leq t$ and $\boldsymbol{y} < 0$ (when I write this for a vector, I mean it holds for each entry). We also know that the graph of non-zero entries in each $B_i$ is connected, and that each $C_i$ is non-positive, and has at least one non-zero entry (otherwise the graph $G$ would be disconnected).

We will now prove that the smallest eigenvalue of $B_i$ is smaller than $\lambda_k$. We know that

$$B_i \boldsymbol{x}_i + C_i \boldsymbol{y} = \lambda_k \boldsymbol{x}_i.$$

As each entry in $C_i$ is non-positive and $\boldsymbol{y}$ is strictly negative, each entry of $C_i\boldsymbol{y}$ is non-negative and some entry of $C_i\boldsymbol{y}$ is positive. Thus, $\boldsymbol{x}_i$ cannot be all zeros,

$$B_i \boldsymbol{x}_i = \lambda_k \boldsymbol{x}_i - C_i \boldsymbol{y} \leq \lambda_k \boldsymbol{x}_i$$

and

$$\boldsymbol{x}_i^T B_i \boldsymbol{x}_i \leq \lambda_k \boldsymbol{x}_i^T \boldsymbol{x}_i.$$

If $\boldsymbol{x}_i$ has *any* zero entries, then the Perron-Frobenius theorem tells us that $\boldsymbol{x}_i$ cannot be an eigenvector of smallest eigenvalue, and so the smallest eigenvalue of $B_i$ is less than $\lambda_k$. On the other hand, if $\boldsymbol{x}_i$ is strictly positive, then $\boldsymbol{x}_i^T C_i \boldsymbol{y} > 0$, and

$$\boldsymbol{x}_i^T B_i \boldsymbol{x}_i = \lambda_k \boldsymbol{x}_i^T \boldsymbol{x}_i - \boldsymbol{x}_i^T C_i \boldsymbol{y} < \lambda_k \boldsymbol{x}_i^T \boldsymbol{x}_i.$$

Thus, the matrix

$$\begin{bmatrix} B_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & B_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & B_t \end{bmatrix}$$

has at least $t$ eigenvalues less than $\lambda_k$. By the eigenvalue interlacing theorem, this implies that $L_G$ has at least $t$ eigenvalues less than $\lambda_k$. We may conclude that $t$, the number of connected components of $G(W_k)$, is at most $k-1$. □

We remark that Fiedler actually proved a somewhat stronger theorem. He showed that the same holds for

$$W = \{i : \boldsymbol{v}_k(i) \geq t\},$$

for every $t \leq 0$.

This theorem breaks down if we instead consider the set

$$W = \{i : \boldsymbol{v}_k(i) > 0\}.$$

The star graphs provide counter-examples.

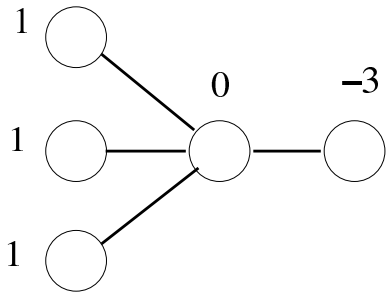



Figure 5.1: The star graph on 5 vertices, with an eigenvector of $\lambda_2 = 1$.

## 5.6 The second Laplacian eigenvalue

The most important eigenvalue will be $\lambda_2$. It is the answer to many questions about graphs, and will entertain us for a few weeks of this course. Let me begin to tell you why.

Recall that $\lambda_2 = 0$ if and only if a graph is disconnected. Fiedler [Fie73] observed that as $\lambda_2$ becomes further from 0, a graph becomes better connected. We will see many versions of this statement, capturing varying ways of measuring connectivity. For the first, we consider the boundary of a set of vertices,

$$\delta(S) = \{(i,j) \in E : i \in S \text{ and } j \notin S\}.$$

**Theorem 5.6.1.** *Let $G = (V, E)$ be a graph and let $L_G$ be its Laplacian matrix. Let $S \subset V$ and set $\sigma = |S| / |V|$. Then,*

$$|\delta(S)| \geq \lambda_2 |S| (1 - \sigma).$$

*Proof.* Recall that

$$\lambda_2 = \min_{\boldsymbol{v}: \boldsymbol{v}^T \mathbf{1} = 0} \frac{\boldsymbol{v}^T L_G \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}},$$

for for every non-zero $\boldsymbol{v}$ orthogonal to $\mathbf{1}$,

$$\boldsymbol{v}^T L_G \boldsymbol{v} \geq \lambda_2 \boldsymbol{v}^T \boldsymbol{v}.$$

To apply this bound, we need to construct a vector related to the set $S$. A natural choice is $\chi_S$. In fact, we have

$$\chi_S^T L_G \chi_S = \sum_{(i,j)\in E} (\chi_S(i) - \chi_S(j))^2 = |\delta(S)|\,.$$

However, $\chi_S$ is not orthogonal to $\mathbf{1}$. To fix this, use

$$\boldsymbol{v} = \chi_S - \sigma\mathbf{1}.$$

We have $\boldsymbol{v}^T\mathbf{1} = 0$, and as $L_G\mathbf{1} = \mathbf{0}$

$$\boldsymbol{v}^T L_G \boldsymbol{v} = \chi_S^T L_G \chi_S = |\delta(S)|\,.$$

To finish the proof, we compute

$$\boldsymbol{v}^T\boldsymbol{v} = |S|\,(1-\sigma)^2 + (|V| - |S|)\sigma^2 = |S|\,(1 - 2\sigma + \sigma^2) + |S|\,\sigma - |S|\,\sigma^2 = |S|\,(1-\sigma).$$

□

This theorem says that if $\lambda_2$ is big, then $G$ is very well connected: the boundary of every small set of vertices is at least $\lambda_2$ times something just slightly smaller than the number of vertices in the set.

This lemma also provides an easy technique for proving upper bounds on $\lambda_2$. Next lecture, we will see techniques for proving lower bonds on $\lambda_2$.

For now, I'd like to show you that there are interesting graphs with large $\lambda_2$.

## 5.7 The Hypercube

The hypercube graph is the graph with vertex set $\{0,1\}^d$, with edges between vertices whose names differ in exactly one bit. The hypercube may also be expressed as the product of the one-edge graph with itself $d-1$ times, with the proper definition of graph product.

**Definition 5.7.1.** *Let $G = (V, E)$ and $H = (W, F)$ be graphs. Then $G \times H$ is the graph with vertex set $V \times W$ and edge set*

$$\Big((v_1, w), (v_2, w)\Big) \text{ where } (v_1, v_2) \in E \text{ and}$$
$$\Big((v, w_1), (v, w_2)\Big) \text{ where } (w_1, w_2) \in F.$$

Let $G = (\{0,1\}, \{(0,1)\})$, and let $H_d$ be the $d$-dimensional hypercube graph. You should check that $H_1 = G$ and that $H_d = H_{d-1} \times G$.

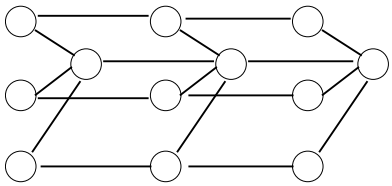


Figure 5.2: The product of a star graph on 4 vertices with a path on 3.

**Theorem 5.7.2.** *Let $G = (V, E)$ and $H = (W, F)$ be graphs with Laplacian eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$, and eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$, respectively. Then, for each $1 \leq i \leq n$ and $1 \leq j \leq m$, $G \times H$ has an eigenvector $\boldsymbol{z}$ of eigenvalue $\lambda_i + \mu_j$ such that*

$$\boldsymbol{z}(v, w) = \boldsymbol{x}_i(v)\boldsymbol{y}_j(w).$$

*Proof.* To see that this eigenvector has the proper eigenvalue, let $L$ denote the Laplacian of $G \times H$, $d_v$ the degree of node $v$ in $G$, and $e_w$ the degree of node $w$ in $H$. We can then verify that

$$\begin{aligned}
(L\boldsymbol{z})(v, w) &= (d_v + e_w)\boldsymbol{x}_i(v)\boldsymbol{y}_j(w) - \sum_{(v,v_2)\in E} \boldsymbol{x}_i(v_2)\boldsymbol{y}_j(w) - \sum_{(w,w_2)\in F} \boldsymbol{x}_i(v)\boldsymbol{y}_j(w_2) \\
&= \left[(d_v)\boldsymbol{x}_i(v)\boldsymbol{y}_j(w) - \sum_{(v,v_2)\in E} \boldsymbol{x}_i(v_2)\boldsymbol{y}_j(w)\right] + \left[(e_w)\boldsymbol{x}_i(v)\boldsymbol{y}_j(w) - \sum_{(w,w_2)\in F} \boldsymbol{x}_i(v)\boldsymbol{y}_j(w_2)\right] \\
&= \boldsymbol{y}_j(w)\left(d_v\boldsymbol{x}_i(v) - \sum_{(v,v_2)\in E} \boldsymbol{x}_i(v_2)\right) + \boldsymbol{x}_i(v)\left(e_w\boldsymbol{y}_j(w) - \sum_{(w,w_2)\in F} \boldsymbol{y}_j(w_2)\right) \\
&= \boldsymbol{y}_j(w)\lambda_i\boldsymbol{x}_i(v) + \boldsymbol{x}_i(v)\mu_j\boldsymbol{y}_j(w) \\
&= (\lambda_i + \mu_j)(\boldsymbol{x}_i(v)\boldsymbol{y}_j(w)).
\end{aligned}$$

□

As the non-zero eigenvector of $G$ is $(1, -1)$ and has eigenvalue 2, we see that $H_d$ has eigenvalue $2k$ with multiplicity $\binom{d}{k}$, for $0 \leq k \leq d$. Using the above theorem, you should also confirm that the eigenvectors of $H_d$ are given by the functions

$$\boldsymbol{v}_a(b) = (-1)^{a^T b},$$

where $a \in \{0, 1\}^d$, and we view vertices $b$ as length-$d$ vectors of zeros and ones. The eigenvalue of which $\boldsymbol{v}_a$ is an eigenvector is the number of ones in $a$.

Using Theorem 5.6.1 and our knowledge of the eigenvalues of the hypercube, we can immediately prove the following isoperimetric theorem for the hypercube.

**Corollary 5.7.3.** *Let $S$ be a subset of $\{0, 1\}^d$ of size at most $2^{d-1}$. Then,*

$$|\delta(S)| \geq |S|.$$

It is possible to prove this by more concrete combinatorial means. But, this proof is simpler.

## References

[Fie73] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(98):298–305, 1973.

[Fie75] M. Fiedler. A property of eigenvectors of nonnegative symmetric matrices and its applications to graph theory. *Czechoslovak Mathematical Journal*, 25(100):618–633, 1975.