Spectral Graph Theory Lecture 6

Graphic Inequalities

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6.1 Overview

I have two goals in this lecture. The first is to give you a technique for proving lower bounds on λ_2. The second is to show you the technique. The technique is cool: it provides a way of proving inequalities on graphs, which we will later use to define what it means for one graph to approximate another.

6.2 Bounds on λ_2

The Courant-Fischer Theorem provides a simple way of proving upper bounds on λ_2. Recall

$$\lambda_2 = \min_{\boldsymbol{v}:\boldsymbol{v}^T\mathbf{1}=0} \frac{\boldsymbol{v}^T L \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}.$$

So, every vector $\boldsymbol{v}$ orthogonal to $\mathbf{1}$ provides an upper bound on λ_2:

$$\lambda_2 \leq \frac{\boldsymbol{v}^T L \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}.$$

When we use a vector $\boldsymbol{v}$ in this way, we call it a *test vector*.

The Courant-Fischer theorem is not as helpful when we want to prove lower bounds on λ_2. To prove lower bounds, we need the form with a maximum on the outside, which gives

$$\lambda_2 \geq \max_{S:\dim(S)=n-1} \min_{v \in S} \frac{\boldsymbol{v}^T L \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}.$$

This is not too helpful, as it is difficult to prove lower bounds on

$$\min_{v \in S} \frac{\boldsymbol{v}^T L \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}$$

over a space S of large dimension. So, we need a new technique.

6.3 Graphic Inequalities

I begin by recalling an extremely useful piece of notation that is used in the Optimization community. For a symmetric matrix A, we write

$$A \succcurlyeq 0$$

if A is positive semidefinite. That is, if

$$\boldsymbol{v}^T A \boldsymbol{v} \geq 0,$$

for all $\boldsymbol{v}$. We similarly write

$$A \succcurlyeq B$$

if

$$\boldsymbol{v}^T A \boldsymbol{v} \geq \boldsymbol{v}^T B \boldsymbol{v}$$

for all $\boldsymbol{v}$. This is the same as

$$A - B \succcurlyeq 0.$$

The relation $\preccurlyeq$ is an example of a *partial order*. It applies to some pairs of symmetric matrices, while others are incomparable. But, for all pairs to which it does apply, it acts like an order. For example, we have

$$A \succcurlyeq B \text{ and } B \succcurlyeq C \text{ implies } A \succcurlyeq C,$$

and

$$A \succcurlyeq B \text{ implies } A + C \succcurlyeq B + C,$$

for symmetric matrices A, B and C.

I find it convenient to overload this notation by defining it for graphs as well. Thus, I'll write

$$G \succcurlyeq H$$

if $L_G \succcurlyeq L_H$. For example, if $G = (V, E)$ is a graph and $H = (V, F)$ is a subgraph of G, then

$$L_G \succcurlyeq L_H.$$

To see this, recall that the Laplacian of a graph can be expressed as the sum of the Laplacians of its edges. As $F \subseteq E$, we get

$$L_G = \sum_{e \in E} L_e = \sum_{e \in F} L_e + \sum_{e \in E - F} L_e \succcurlyeq \sum_{e \in F} L_e = L_H,$$

as

$$\sum_{e \in E - F} L_e \succcurlyeq 0.$$

In this proof, I have used the notation L_e to indicate the Laplacian consisting of the graph containing just the edge e.

This notation is most powerful when we consider some multiple of a graph. Thus, I could write

$$G \succcurlyeq c \cdot H,$$

for some $c > 0$. What is $c \cdot H$? It is the same graph as H, but the weight of every edge is multiplied by c.

Using the Courant-Fischer Theorem, we can prove

Lemma 6.3.1. *If G and H are graphs such that*

$$G \succcurlyeq c \cdot H,$$

then

$$\lambda_k(G) \geq c\lambda_k(H),$$

for all k.

Proof. The Courant-Fischer Theorem tells us that

$$\lambda_k(G) = \min_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \max_{\boldsymbol{x} \in S} \frac{\boldsymbol{x}^T L_G \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} \geq \min_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \max_{\boldsymbol{x} \in S} c \frac{\boldsymbol{x}^T L_H \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} = c \min_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \max_{\boldsymbol{x} \in S} \frac{\boldsymbol{x}^T L_H \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} = c\lambda_k(H).$$

□

This lemma provides an easy way of bounding how much the eigenvalues of a graph can change if we change the weights on some of its edges.

Lemma 6.3.2. *Let $G = (V, E, w)$ and $H = (V, E, z)$ be two graphs that differ only in their edge weights. Then*

$$G \succcurlyeq \min_{e \in E} \frac{w(e)}{z(e)} H.$$

Proof. Recall that the Laplacian of a graph may be expressed as the sum of the Laplacians of its edges. So,

$$L_G = \sum_{e \in E} w(e) L_e = \sum_{e \in E} \frac{w(e)}{z(e)} z(e) L_e \geq \left(\min_{e \in E} \frac{w(e)}{z(e)} \right) \sum_{e \in E} z(e) L_e = \left(\min_{e \in E} \frac{w(e)}{z(e)} \right) L_H.$$

□

6.4 Approximations of Graphs

An idea that we will use in later lectures is that one graph approximates another if their Laplacian quadratic forms are similar. For example, we will say that H is a c-approximation of G if

$$cH \succcurlyeq G \succcurlyeq H.$$

Since I really care about graph structure more than constants, I will say that H is a c-approximation of G if any multiple of H is a c-approximation of G.

Definition 6.4.1. *A graph H is a c-approximation of G if there exists a positive t for which*

$$cH \succcurlyeq tG \succcurlyeq H.$$

Surprising approximations exist. For example, expander graphs are very sparse approximations of the complete graph. For example, the following is known.

Theorem 6.4.2. *For every $\epsilon > 0$, there exists a $d > 0$ such that for all sufficiently large n there is a d-regular graph G_n that is a $(1+\epsilon)$-approximation of K_n.*

These graphs have many fewer edges than the complete graphs!

6.5 The Path Inequality

Now, the question is: how do we prove that $G \succcurlyeq c \cdot H$ for some graph G and H? Not too many ways are known. We'll do it by proving some inequalities of this form for some of the simplest graphs, and then extending them to more general graphs. For example, we will prove

$$(n-1) \cdot P_n \succcurlyeq G_{1,n}. \tag{6.1}$$

That is, $n-1$ times the path of length $n-1$ from vertex 1 to n is greater than the edge from 1 to n.

The following very simple proof of this inequality was discovered by Sam Daitch.

Lemma 6.5.1.

$$(n-1) \cdot P_n \succcurlyeq G_{1,n}.$$

Proof. We need to show that for every $\boldsymbol{x} \in \mathbb{R}^n$,

$$(n-1) \sum_{i=1}^{n-1} (\boldsymbol{x}(i+1) - \boldsymbol{x}(i))^2 \geq (\boldsymbol{x}(n) - \boldsymbol{x}(1))^2.$$

For $1 \leq i \leq n-1$, set

$$\boldsymbol{\delta}(i) = \boldsymbol{x}(i+1) - \boldsymbol{x}(i).$$

The inequality we need to prove then becomes

$$(n-1) \sum_{i=1}^{n-1} \boldsymbol{\delta}(i)^2 \geq \left(\sum_{i=1}^{n-1} \boldsymbol{\delta}(i) \right)^2.$$

But, this is just the Cauchy-Schwartz inequality. I'll remind you that Cauchy-Schwartz just follows from the fact that the inner product of two vectors is at most the product of their norms:

$$(n-1) \sum_{i=1}^{n-1} \boldsymbol{\delta}(i)^2 = \|\mathbf{1}_{n-1}\|^2 \|\boldsymbol{\delta}\|^2 = (\|\mathbf{1}_{n-1}\| \|\boldsymbol{\delta}\|)^2 \geq \left(\mathbf{1}_{n-1}^T \boldsymbol{\delta}\right)^2 = \left(\sum_{i=1}^{n-1} \boldsymbol{\delta}(i) \right)^2.$$

$\square$

While I won't cover it in lecture, I will also state the version of this inequality for weighted paths.

Lemma 6.5.2. *Let $w_1, \ldots, w_{n-1}$ be positive. Then*

$$G_{1,n} \preccurlyeq \left(\sum_{i=1}^{n-1} \frac{1}{w_i} \right) \sum_{i=1}^{n-1} w_i G_{i,i+1}.$$

Proof. Let $\boldsymbol{x} \in \mathbb{R}^n$ and set $\boldsymbol{\delta}(i)$ as in the proof of the previous lemma. Now, set

$$\boldsymbol{\gamma}(i) = \boldsymbol{\delta}(i)\sqrt{w_i}.$$

Let $\boldsymbol{w}^{-1/2}$ denote the vector for which

$$\boldsymbol{w}^{-1/2}(i) = \frac{1}{\sqrt{w_i}}.$$

Then,

$$\sum_i \boldsymbol{\delta}(i) = \boldsymbol{\gamma}^T \boldsymbol{w}^{-1/2},$$

$$\left\| \boldsymbol{w}^{-1/2} \right\|^2 = \sum_i \frac{1}{w_i},$$

and

$$\|\boldsymbol{\gamma}\|^2 = \sum_i \boldsymbol{\delta}(i)^2 w_i.$$

So,

$$\begin{aligned} \boldsymbol{x}^T L_{G_{1,n}} \boldsymbol{x} &= \left(\sum_i \boldsymbol{\delta}(i) \right)^2 = \left(\boldsymbol{\gamma}^T \boldsymbol{w}^{-1/2} \right)^2 \\ &\leq \left(\|\boldsymbol{\gamma}\| \left\| \boldsymbol{w}^{-1/2} \right\| \right)^2 = \left(\sum_i \frac{1}{w_i} \right) \sum_i \boldsymbol{\delta}(i)^2 w_i = \left(\sum_i \frac{1}{w_i} \right) \boldsymbol{x}^T \left(\sum_{i=1}^{n-1} w_i L_{G_{i,i+1}} \right) \boldsymbol{x}. \end{aligned}$$

□

6.5.1 Bounding λ_2 of a Path Graph

Even though we already know all the eigenvalues of an unweighted path graph, I am going to take a moment to demonstrate how one can use these techniques to easily get bounds on λ_2 of a path graph. First, let's use a test vector to get an upper bound.

Consider the vector $\boldsymbol{x}$ such that $\boldsymbol{x}(i) = (n+1) - 2i$, for $1 \leq i \leq n$. This vector satisfies $\boldsymbol{x} \perp \mathbf{1}$, so

$$\begin{aligned} \lambda_2(P_n) &\leq \frac{\sum_{1 \leq i < n} (x(i) - x(i+1))^2}{\sum_i x(i)^2} \\ &= \frac{\sum_{1 \leq i < n} 2^2}{\sum_i (n+1-2i)^2} \\ &= \frac{4(n-1)}{(n+1)n(n-1)/3} \\ &= \frac{12}{n(n+1)}. \end{aligned}$$

So, we can easily obtain an upper bound on $\lambda(P_n)$ that is of the right order of magnitude.

To prove a lower bound on $\lambda_2(P_n)$, we will prove that some multiple of the path is at least the complete graph. To this end, recall that

$$L_{K_n} = \sum_{i<j} L_{G_{i,j}},$$

and that

$$\lambda_2(K_n) = n.$$

For every edge (i, j) in the complete graph, we apply the only inequality available in the path:

$$G_{i,j} \preccurlyeq (j-i) \sum_{k=i}^{j-1} G_{k,k+1} \preccurlyeq (j-i) P_n.$$

Summing this inequality over all edges $(i, j) \in K_n$ gives

$$K_n = \sum_{i<j} G_{i,j} \preccurlyeq \sum_{i<j} (j-i) P_n.$$

To finish the proof, we compute

$$\sum_{1 \leq i < j \leq n} (j-i) = \sum_{k=1}^{n-1} k(n-k) = n(n+1)(n-1)/6.$$

So,

$$\frac{n(n+1)(n-1)}{6} \cdot L_{P_n} \succcurlyeq L_{K_n}.$$

Applying Lemma 6.3.1, we obtain

$$\lambda_2(P_n) \geq \frac{6}{(n+1)(n-1)}.$$

This only differs from our lower bound by a factor of 2.

6.5.2 The Complete Binary Tree

Let's do the same analysis with the complete binary tree.

The complete binary tree on $n = 2^d - 1$ nodes, T_n, is the graph with edges of the form $(i, 2i)$ and $(i, 2i+1)$ for $i < n/2$. Pictorially, these graphs look like this:

Let's first upper bound $\lambda_2(B_n)$ by constructing a test vector x. Set $x(1) = 0$, $x(2) = 1$, and $x(3) = -1$. Then, for every vertex u that we can reach from node 2 without going through node 1, we set $x(u) = 1$. For all the other nodes, we set $x(u) = -1$.

We then have

$$\lambda_2 \leq \frac{\sum_{(i,j) \in T_n} (x_i - x_j)^2}{\sum_i x_i^2} = \frac{(x_1 - x_2)^2 + (x_1 - x_3)^2}{n-1} = 2/(n-1).$$

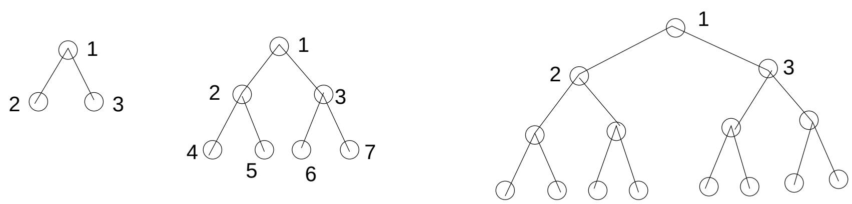

Figure 6.1: T_3, T_7 and T_{15}. Node 1 is at the top, 2 and 3 are its children. Some other nodes have been labeled as well.

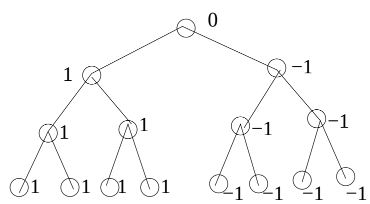

Figure 6.2: The test vector we use to upper bound $\lambda_2(T_{15})$.

We will again prove a lower bound by comparing T_n to the complete graph. For each edge $(i,j) \in K_n$, let $T_n(i,j)$ denote the unique path in T from i to j. This path will have length at most $2\log_2 n$. So, we have

$$K_n = \sum_{i<j} G_{i,j} \preccurlyeq \sum_{i<j} (2\log_2 n) T_n(i,j) \preccurlyeq \sum_{i<j} (2\log_2 n) T_n = \binom{n}{2} (2\log_2 n) T_n.$$

So, we obtain the bound

$$\binom{n}{2} (2\log_2 n) \lambda_2(T_n) \geq n,$$

which implies

$$\lambda_2(T_n) \geq \frac{1}{(n-1)\log_2 n}.$$

In the problem set, I will ask you to improve this lower bound to $1/cn$ for some constant c.