Spectral Graph Theory Lecture 10

Properties of Expanders

Daniel A. Spielman October 2, 2009

10.1 Expanders as Approximations of the Complete Graph

Last lecture, we defined an expander graph to be a d-regular graph whose adjacency matrix eigenvalues satisfy

$$|\alpha_i| \leq \epsilon d, \tag{10.1}$$

for $i \geq 2$, for some small ϵ. We will now see that graphs satisfying this condition are very good approximations of the complete graph.

I say that a graph G is an ϵ-approximation of a graph H if

$$(1-\epsilon)H \preccurlyeq G \preccurlyeq (1+\epsilon)H,$$

where I recall that I say $H \preccurlyeq G$ if for all $\boldsymbol{x}$

$$\boldsymbol{x}^T L_H \boldsymbol{x} \leq \boldsymbol{x}^T L_G \boldsymbol{x}.$$

Let G be a graph whose adjacency eigenvalues satisfy (10.1). As its Laplacian eigenvalues satisfy $\lambda_i = d - \alpha_i$, all non-zero eignevalues of L_G are between $(1-\epsilon)d$ and $(1+\epsilon)d$. This means that for all $\boldsymbol{x}$ orthogonal to the all-1s vector,

$$(1-\epsilon)d\boldsymbol{x}^T\boldsymbol{x} \leq \boldsymbol{x}^T L_G \boldsymbol{x} \leq (1+\epsilon)d\boldsymbol{x}^T\boldsymbol{x}.$$

You may verify this by Courant-Fischer, or by expanding $\boldsymbol{x}$ in an eigenbasis.

On the other hand, for the complete graph K_n, we know that all $\boldsymbol{x}$ orthogonal to the all-1s vector satisfy

$$\boldsymbol{x}^T L_{K_n} \boldsymbol{x} = n\boldsymbol{x}^T\boldsymbol{x}.$$

Let H be the graph

$$H = \frac{d}{n} K_n,$$

so

$$\boldsymbol{x}^T L_H \boldsymbol{x} = d\boldsymbol{x}^T\boldsymbol{x}.$$

Thus, G is an ϵ-approximation of H.

This tells us that $G - H$ is a matrix of small norm. Observe that

$$(1-\epsilon)H \preccurlyeq G \preccurlyeq (1+\epsilon)H \quad \text{implies} \quad -\epsilon H \preccurlyeq G - H \preccurlyeq \epsilon H.$$

As G and H are symmetric, and all eigenvalues of ϵH are 0 or d, we may infer

$$\|L_G - L_H\| \leq \epsilon d. \tag{10.2}$$

10.2 Quasi-Random Properties of Expanders

There are many ways in which expander graphs act like random graphs. In fact, one can prove that a random d-regular graph is an expander graph with reasonably high probability.

We also know that all sets of vertices in an expander graph act like random sets of vertices. To make this precise, imagine creating a random set $S \subset V$ by including each vertex in S independently with probabilty α. How many edeges do we expect to find between vertices in S? Well, for every edge (u, v), the probability that $u \in S$ is α and the probability that $v \in S$ is α, so the probability that both endpoints are in S is α^2. So, we expect an α^2 fraction of the edges to go between vertices in S. We will show that this is true *for all sufficiently large sets* S in an expander.

In fact, we will prove a stronger version of this statement for two sets S and T. Imagine including each vertex in S independently with probability α and each vertex in T with probability β. We allow vertices to belong to both S and T. For how many ordered pairs $(u, v) \in E$ do we expect to have $u \in S$ and $v \in T$? Obviously, it should hold for an $\alpha\beta$ fraction of the pairs.

For a graph $G = (V, E)$, define

$$\vec{E}(S, T) = \{(u, v) : u \in S, v \in T, (u, v) \in E\}.$$

We have put the arrow above the E in the definition, because we are considering ordered pairs of vertices. When S and T are disjoint

$$\left|\vec{E}(S, T)\right|$$

is precisely the number of edges between S and T, while

$$\left|\vec{E}(S, S)\right|$$

counts every edge inside S twice.

The following bound is a slight extension by Beigel, Margulis and Spielman [BMS93] of a bound originally proved by Alon and Chung [AC88].

Theorem 10.2.1. *Let $G = (V, E)$ be a d-regular graph that ϵ-approximates $\frac{d}{n}K_n$. Then, for every $S \subseteq V$ and $T \subseteq V$,*

$$\left|\left|\vec{E}(S, T)\right| - \alpha\beta dn\right| \leq \epsilon dn\sqrt{(\alpha - \alpha^2)(\beta - \beta^2)},$$

where $|S| = \alpha n$ and $|T| = \beta n$.

Observe that when α and β are greater than ϵ, the term on the right is less than $\alpha\beta dn$.

In class, we will just prove this in the case that S and T are disjoint.

Proof. The first step towards the proof is to observe

$$\chi_S^T L_G \chi_T = d\,|S \cap T| - \left|\vec{E}(S, T)\right|.$$

Let $H = \frac{d}{n} K_n$. As G is a good approximation of H, let's compute

$$\chi_S^T L_H \chi_T = \chi_S^T \left(dI - \frac{d}{n} J \right) \chi_T = d \left| S \cap T \right| - \frac{d}{n} \left| S \right| \left| T \right| = d \left| S \cap T \right| - \alpha\beta dn.$$

So,

$$\left| \left| \vec{E}(S,T) \right| - \alpha\beta dn \right| = \left| \chi_S^T L_G \chi_T - \chi_S^T L_H \chi_T \right|.$$

As

$$\left\| L_G - L_H \right\| \leq \epsilon d,$$

$$\begin{aligned}
\chi_S^T L_H \chi_T - \chi_S^T L_G \chi_T &= \chi_S^T (L_H - L_G) \chi_T \\
&\leq \left\| \chi_S \right\| \left\| (L_H - L_G) \chi_T \right\| \\
&\leq \left\| \chi_S \right\| \left\| L_H - L_G \right\| \left\| \chi_T \right\| \\
&\leq \epsilon d \left\| \chi_S \right\| \left\| \chi_T \right\| \\
&= \epsilon dn \sqrt{\alpha\beta}.
\end{aligned}$$

This is almost as good as the bound we are trying to prove. To prove the claimed bound, we should orthogonalize χ_S and χ_T with respect to the all-1's vector. We then get

$$\chi_S^T (L_H - L_G) \chi_T = (\chi_S - \alpha \mathbf{1})^T (L_H - L_G)(\chi_T - \beta \mathbf{1}),$$

while

$$\left\| \chi_S - \alpha \mathbf{1} \right\| \left\| \chi_T - \alpha \mathbf{1} \right\| = n \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}.$$

So, we may conclude

$$\left| \left| \vec{E}(S,T) \right| - \alpha\beta dn \right| \leq \epsilon dn \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}.$$

□

We remark that when S and T are disjoint, the same proof goes through even if G is irregular and weighted if we replace $\vec{E}(S,T)$ with

$$w(S,T) = \sum_{(u,v) \in E, u \in S, v \in T} w(u,v).$$

We only need the fact that G ϵ-approximates $\frac{d}{n} K_n$. See [BSS09] for details.

10.3 Vertex Expansion

The reason for the name *expander graph* is that small sets of vertices in expander graphs have unusually large numbers of neighbors. For $S \subset V$, let $N(S)$ denote the set of vertices that are neighbors of vertices in S. The following theorem, called Tanner's Theorem, provides a lower bound on the size of $N(S)$.

Theorem 10.3.1 ([Tan84]). *Let $G = (V, E)$ be a d-regular graph on n vertices that ϵ-approximates $\frac{d}{n} K_n$. Then, for all $S \subseteq V$,*

$$|N(S)| \geq \frac{|S|}{\epsilon^2(1-\alpha)+\alpha},$$

where $|S| = \alpha n$.

Note that when α is much less than ϵ^2, the term on the right is approximately $|S|/\epsilon^2$, which can be much larger than $|S|$. We will derive Tanner's theorem from Theorem 10.2.1.

Proof. Let $R = N(S)$ and let $T = V - R$. Then, there are no edges between S and T. Let $|T| = \beta n$ and $|R| = \gamma n$, so $\gamma = 1 - \beta$. By Theorem 10.2.1, it must be the case that

$$\alpha\beta dn \leq \epsilon dn \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}.$$

The lower bound on γ now follows by re-arranging terms. For example, we have

$$\begin{aligned}
\alpha^2\beta^2 &\leq \epsilon^2(\alpha-\alpha^2)(\beta-\beta^2) && \text{implies}\\
\alpha\beta &\leq \epsilon^2(1-\alpha)(1-\beta) && \text{implies}\\
\frac{\beta}{1-\beta} &\leq \frac{\epsilon^2(1-\alpha)}{\alpha} && \text{implies}\\
\frac{1-\gamma}{\gamma} &\leq \frac{\epsilon^2(1-\alpha)}{\alpha} && \text{implies}\\
\frac{1}{\gamma} &\leq \frac{\epsilon^2(1-\alpha)+\alpha}{\alpha} && \text{implies}\\
\gamma &\geq \frac{\alpha}{\epsilon^2(1-\alpha)+\alpha}.
\end{aligned}$$

□

If instead of $N(S)$ we consider $N(S) - S$, then T and S are disjoint, so the same proof goes through for weighted, irregular graphs that ϵ-approximate $\frac{d}{n} K_n$.

10.4 How well can a graph approximate the complete graph?

Consider applying Tanner's Theorem with $S = \{v\}$ for some vertex v. As v has exactly d neighbors, we find

$$\epsilon^2(1-1/n) + 1/n \geq 1/d,$$

from which we see that ϵ must be at least $1/\sqrt{d + d^2/n}$, which is essentially $1/\sqrt{d}$. But, how small can it be?

The Ramanujan graphs, constructed by Margulis [Mar88] and Lubotzky, Phillips and Sarnak [LPS88] achieve

$$\epsilon \leq \frac{2\sqrt{d-1}}{d}.$$

We will see that if we keep d fixed while we let n grow, ϵ cannot exceed this bound in the limit. We will prove an upper bound on ϵ by constructing a suitable test function.

As a first step, choose two vertices v and u in V whose neighborhoods to do not overlap. Consider the vector $\boldsymbol{x}$ defined by

$$\boldsymbol{x}(i) = \begin{cases} 1 & \text{if } i = u, \\ 1/\sqrt{d} & \text{if } i \in N(u), \\ -1 & \text{if } i = v, \\ -1/\sqrt{d} & \text{if } i \in N(v), \\ 0 & \text{otherwise.} \end{cases}$$

Now, compute the Rayleigh quotient with respect to $\boldsymbol{x}$. The numerator is the sum over all edges of the squares of differences accross the edges. This gives $(1 - 1/\sqrt{d})^2$ for the edges attached to u and v, and $1/d$ for the edges attached to $N(u)$ and $N(v)$ but not to u or v, for a total of

$$2d(1 - 1/\sqrt{d})^2 + 2d(d-1)/d = 2\left(d - 2\sqrt{d} + 1 + (d-1)\right) = 2\left(2d - 2\sqrt{d}\right).$$

On the other hand, the denominator is 4, so we find

$$\frac{\boldsymbol{x}^T L \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} = d - \sqrt{d}.$$

If we use instead the vector

$$\boldsymbol{y}(i) = \begin{cases} 1 & \text{if } i = u, \\ -1/\sqrt{d} & \text{if } i \in N(u), \\ -1 & \text{if } i = v, \\ 1/\sqrt{d} & \text{if } i \in N(v), \\ 0 & \text{otherwise,} \end{cases}$$

we find

$$\frac{\boldsymbol{y}^T L \boldsymbol{y}}{\boldsymbol{y}^T \boldsymbol{y}} = d + \sqrt{d}.$$

This is not so impressive, as it merely tells us that $\epsilon \geq 1/\sqrt{d}$, which we already knew. But, we can improve this argument by pushing it further. We do this by modifying it in two ways. First, we extend $\boldsymbol{x}$ to neighborhoods of neighboods of u and v. Second, instead of basing the construction at vertices u and v, we base it at two edges. This way, each vertex has $d-1$ edges to those that are farther away from the centers of the construction.

The following theorem is attributed to A. Nilli [Nil91], but we suspect it was written by N. Alon.

Theorem 10.4.1. *Let G be a d-regular graph containing two edges (u_0, u_1) and (v_0, v_1) that are at distance at least $2k+2$. Then*

$$\lambda_2 \leq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1} - 1}{k+1},$$

and

$$\lambda_n \geq d + 2\sqrt{d-1} - \frac{2\sqrt{d-1} - 1}{k+1}.$$

Proof. Define the following neighborboods.

$$\begin{aligned}
U_0 &= \{u_0, u_1\} \\
U_i &= N(U_{i-1}) - \cup_{j<i} U_j, \quad \text{for } 0 < i \leq k, \\
V_0 &= \{v_0, v_1\} \\
V_i &= N(V_{i-1}) - \cup_{j<i} V_j, \quad \text{for } 0 < i \leq k.
\end{aligned}$$

That is, U_i consists of exactly those vertices at distance i from U_0. Note that there are no edges between any vertices in any U_i and any V_j.

Our test vector for λ_2 will be given by

$$\boldsymbol{x}(a) = \begin{cases} \frac{1}{(d-1)^{-i/2}} & \text{for } a \in U_i \\ -\frac{\beta}{(d-1)^{-i/2}} & \text{for } a \in V_i \\ 0 & \text{otherwise.} \end{cases}$$

We choose β so that $\boldsymbol{x}$ is orthogonal to $\mathbf{1}$.

We now find that the Rayleigh quotient of $\boldsymbol{x}$ with respect to L is at most

$$\frac{\left(\sum_{i=0}^{k-1} |U_i| (d-1) \left(\frac{1-1/\sqrt{d-1}}{(d-1)^{-i/2}}\right)^2 + |U_k| (d-1)^{-k+1}\right) + \beta^2 \left(\sum_{i=0}^{k-1} |V_i| (d-1) \left(\frac{1-1/\sqrt{d-1}}{(d-1)^{-i/2}}\right)^2 + |V_k| (d-1)^{-k+1}\right)}{\sum_{i=0}^{k} |U_i| (d-1)^{-i} + \beta^2 \sum_{i=0}^{k} |V_i| (d-1)^{-i}}$$

By my favorite inequality, it suffices to prove upper bounds on the left and right-hand terms in these fractions. So, consider

$$\frac{\sum_{i=0}^{k-1} |U_i| (d-1) \left(\frac{1-1/\sqrt{d-1}}{(d-1)^{-i/2}}\right)^2 + |U_k| (d-1)^{-k+1}}{\sum_{i=0}^{k} |U_i| (d-1)^{-i}}.$$

For now, let's focus on the numerator,

$$\begin{aligned}
&\sum_{i=0}^{k-1} |U_i| (d-1) \left(\frac{1-1/\sqrt{d-1}}{(d-1)^{-i/2}}\right)^2 + |U_k| (d-1)(d-1)^{-k} \\
&= \sum_{i=0}^{k-1} \frac{|U_i|}{(d-1)^i}(d-2\sqrt{d-1}) + \frac{|U_k|}{(d-1)^k}(d-1) \\
&= \sum_{i=0}^{k-1} \frac{|U_i|}{(d-1)^i}(d-2\sqrt{d-1}) + \frac{|U_k|}{(d-1)^k}(d-2\sqrt{d-1}) + \frac{|U_k|}{(d-1)^k}(2\sqrt{d-1}-1) \\
&= \sum_{i=0}^{k} \frac{|U_i|}{(d-1)^i}(d-2\sqrt{d-1}) + \frac{|U_k|}{(d-1)^k}(2\sqrt{d-1}-1).
\end{aligned}$$

To upper bound the Rayleigh quotient, we observe that the left-most of these terms contributes

$$\frac{\sum_{i=0}^{k} \frac{|U_i|}{(d-1)^i}(d-2\sqrt{d-1})}{\sum_{i=0}^{k} |U_i| (d-1)^{-i}} = d - 2\sqrt{d-1}.$$

To bound the impact of the remaining term,

$$\frac{|U_k|}{(d-1)^k}(2\sqrt{d-1}-1),$$

note that

$$|U_k| \leq (d-1)^{k-i}\,|U_i|\,.$$

So, we have

$$\frac{|U_k|}{(d-1)^k} \leq \frac{1}{k+1}\sum_{i=0}^{k}\frac{|U_i|}{(d-1)^i}.$$

Thus, the last term contributes at most

$$\frac{2\sqrt{d}-1}{k+1}$$

to the Rayleigh quotient.

To prove the lower bound on λ_n, we apply the same argument to the test function

$$\boldsymbol{y}(a) = \begin{cases} (-1)^i \frac{1}{(d-1)^{-i/2}} & \text{for } a \in U_i \\ -(-1)^i \frac{\beta}{(d-1)^{-i/2}} & \text{for } a \in V_i \\ & \\ 0 & \text{otherwise.} \end{cases}$$

□

10.5 Research Question

We still don't know how well general irregular, weighted graphs can approximate the complete graph. If $G = (V, E, w)$ has $dn/2$ edges, one can still show that it cannot be an ϵ-approximation of the complete graph for $\epsilon < 1/\sqrt{d}$ (see [BSS09]). I conjecture that in fact one cannot have $\epsilon < 2\sqrt{d-1}/d$, the same bound for the regular case. Any bound close to this would be publishable.

References

[AC88] Noga Alon and Fan Chung. Explicit construction of linear sized tolerant networks. *Discrete Mathematics*, 72:15–19, 1988.

[BMS93] Richard Beigel, Grigorii Margulis, and Daniel A. Spielman. Fault diagnosis in a small constant number of parallel testing rounds. In *SPAA '93: Proceedings of the fifth annual ACM symposium on Parallel algorithms and architectures*, pages 21–29, New York, NY, USA, 1993. ACM.

[BSS09] Joshua D. Batson, Daniel A. Spielman, and Nikhil Srivastava. Twice-Ramanujan sparsifiers. In *Proceedings of the 41st Annual ACM Symposium on Theory of computing*, pages 255–262, 2009.

[LPS88] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[Mar88] G. A. Margulis. Explicit group theoretical constructions of combinatorial schemes and their application to the design of expanders and concentrators. *Problems of Information Transmission*, 24(1):39–46, July 1988.

[Nil91] A. Nilli. On the second eigenvalue of a graph. *Discrete Math*, 91:207–210, 1991.

[Tan84] R. Michael Tanner. Explicit concentrators from generalized n-gons. *SIAM Journal Alg. Disc. Meth.*, 5(3):287–293, September 1984.