Spectral Graph Theory	Lecture 14
The Simplest Construction of Expanders	
Daniel A. Spielman	October 16, 2009

14.1 Overview

I am going to present the simplest construction of expanders that I have been able to find. By "simplest", I mean optimizing the tradeoff of simplicity of construction with simplicty of analysis. It is inspired by the Zig-Zag product and replacement product constructions presented by Reingold, Vadhan and Wigderson [RVW02].

For those who want the quick description, here it is. Begin with an expander. Take its line graph. Obsever that the line graph is a union of cliques. So, replace each clique by a small expander. We need to improve the expansion slightly, so square the graph. Square one more time. Repeat.

The analysis will be simple because all of the important parts are equalities, which I find easier to understand than inequalities.

14.2 Line Graphs

Our construction will leverage small expanders to make bigger expanders. To begin, we need a way to make a graph bigger and still say something about its spectrum.

We use the *line graph* of a graph. Let $G = (V, E)$ be a graph. The line graph of G is the graph whose vertices are the edges of G in which two are connected if they share an endpoint in G. That is, $\big((u, v), (w, z)\big)$ is an edge of the line graph if one of $\{u, v\}$ is the same as one of $\{w, z\}$. The line graph is often written $L(G)$, but we won't do that in this class so that we can avoid confusion with the Laplacian.

insert picture here

Let G be a d-regular graph with n vertices, and let H be its line graph. As G has $dn/2$ edges, H has $dn/2$ vertices. Each vertex of H, say (u, v), has degree $2(d-1)$: $d-1$ neighbors for the other edges attached to u and $d-1$ for v. In fact, if we just consider one vertex u in V, then all vertices in H of form (u, v) of G will be connected. That is, H contains a d-clique for every vertex in V. We see that each vertex of H is contained in exactly two of these cliques.

Here is the great fact about the spectrum of the line graph.

Lemma 14.2.1. *Let G be a d-regular graph with n vertices, and let H be its line graph. Then the spectrum of the Laplacian of H is the same as the spectum of the Laplacian of G, except that it has $dn/2 - n$ extra eigenvalues of $2(d-1)$.*

Before we prove this lemma, we need to recall an elementary fact about the Laplacian that we neglected to mention in earlier lectures.

14.3 A Factorization of a Laplacian

Recall from Lecture 2 that the Laplacian matrix of an unweighted graph G is given by

$$L_G = D_G - A_G = \sum_{(u,v) \in E} L_{G_{u,v}},$$

where by $L_{G_{u,v}}$ we mean the Laplacian matrix of the graph containing just the edge (u, v). This matrix is zero every where, except on the rows and columns indexed by u and v on which it looks like

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

This latter matrix may be simply factored as

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}.$$

So, if we let $\boldsymbol{\chi}_u$ denote the column vector that is 1 in position u and zero elsewhere, we can write

$$L_{G_{u,v}} = (\boldsymbol{\chi}_u - \boldsymbol{\chi}_v)(\boldsymbol{\chi}_u - \boldsymbol{\chi}_v)^T,$$

and

$$L_G = \sum_{(u,v) \in E} (\boldsymbol{\chi}_u - \boldsymbol{\chi}_v)(\boldsymbol{\chi}_u - \boldsymbol{\chi}_v)^T.$$

Recall that when $\boldsymbol{x}$ and $\boldsymbol{y}$ are column vectors,

$$\boldsymbol{x}\boldsymbol{x}^T + \boldsymbol{y}\boldsymbol{y}^T = \begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^T \\ \boldsymbol{y}^T \end{bmatrix}.$$

So, we can write L_G as the product of a matrix with its transpose. Let U be the n-by-m matrix with rows indexed by vertices and columns indexed by edges with the column correpsonding to edge (u, v) being $\boldsymbol{\chi}_u - \boldsymbol{\chi}_v$. Then,

$$L_G = UU^T.$$

This matrix is called a *signed vertex-edge incidence matrix of* G. Note that we have made an arbitrary choice in our construction of U. We could have chosen $\boldsymbol{\chi}_u - \boldsymbol{\chi}_v$ or $\boldsymbol{\chi}_v - \boldsymbol{\chi}_u$. It doesn't matter which we choose, as long as we make a choice. It factors out when we multiply U by U^T.

14.4 The Spectrum of the Line Graph

Define the matrix $|U|$ to be the matrix obtained by replacing every entry of U by its absolute value. Now, consider $|U|\,|U|^T$. It looks just like the Laplacian, except that all of its off-diagonal entries are 1 instead of -1. So,

$$|U|\,|U|^T = D_G + A_G.$$

We will also consider the matrix $|U|^T |U|$. This is a matrix with $nd/2$ rows and $nd/2$ columns, indexed by edges of G. The entry at the intersection of row (u, v) and column (w, z) is

$$(\boldsymbol{\chi}_u + \boldsymbol{\chi}_v)^T (\boldsymbol{\chi}_w + \boldsymbol{\chi}_z).$$

So, it is 2 if these are the same edge, 1 if they share a vertex, and 0 otherwise. That is

$$|U|^T |U| = 2I_{nd/2} + A_H.$$

Moreover, $|U| |U|^T$ and $|U|^T |U|$ have the same eigenvalues, except that the later matrix has $nd/2-n$ extra eigenvalues of 0.

Proof of Lemma 14.2.1. First, let λ_i be an eigenvalue of L_G. We see that

$$\begin{aligned}
&\lambda_i \text{ is an eigenvalue of } D_G - A_G \implies \\
&d - \lambda_i \text{ is an eigenvalue of } A_G \implies \\
&2d - \lambda_i \text{ is an eigenvalue of } D_G + A_G \implies \\
&2d - \lambda_i \text{ is an eigenvalue of } 2I_{nd/2} + A_H \implies \\
&2(d-1) - \lambda_i \text{ is an eigenvalue of } A_H \implies \\
&\lambda_i \text{ is an eigenvalue of } D_H - A_H.
\end{aligned}$$

Of course, this last matrix is the Laplacian matrix of H. We can similarly show that the extra $dn/2 - n$ zero eigenvalues of $2I_{nd/2} + A_H$ become $2(d-1)$ in L_H. □

While the line graph operation preserves λ_2, it causes the degree of the graph to grow. So, we are going to need to do more than just take line graphs to construct expanders.

We will analyze the graphs that appear in our construction by keeping track of the ratio of their second-smallest Laplacian eigenvalues to their degree: λ_2/d. We will call this quantity the *spectral ratio* of a graph. Because of the nature of our construction, we will not need to worry about λ_n, as it is guaranteed to be close to the degree in the end. Our construction will produce an infinite family of d-regular graphs with spectral ratio bounded below by some absolute constant $\beta > 0$. We will do this for small β, as the analysis for large β is trickier.

We see that the spectral ratio of the line graph of G is approximately half that of G. But, the line graph has more vertices.

14.5 Approximations of Line Graphs

Our next step will be to construct approximations of line graphs. We already know how to approximate complete graphs: we use expanders. As line graphs are sums of complete graphs, we will approximate them by sums of expanders. That is, we replace each clique in the line graph by an expander on d vertices.

Let G be a d-regular graph and let K be a graph on d vertices (we will use a low-degree expander). We define the graph

$$G \textcircled{L} K$$

to be the graph obtained by forming the edge graph of G, H, and then replacing every d-clique in H by a copy of K. Actually, this does not uniquely define $G \textcircled{L} K$, as there are many ways to replace a d-clique by a copy of K. But, any choice will work. Note that every vertex of $G \textcircled{L} K$ has degree $2k$.

Lemma 14.5.1. *Let G be a d-regular line graph, and let K be a k-regular graph on d vertices that ϵ-approximates $\frac{k}{d} K_d$. Let H be the line graph of G. Then,*

$$(1-\epsilon) G \textcircled{L} K \preccurlyeq \frac{k}{d} H \preccurlyeq (1+\epsilon) G \textcircled{L} K$$

Proof. As H is a sum of d-cliques, let $H_1, \ldots, H_n$ be those d-cliques. So,

$$L_H = \sum_{i=1}^{n} L_{H_i}.$$

Let K_i be the graph obtained by replacing H_i with a copy of K, on the same set of vertices. To prove the lower bound let $\boldsymbol{x}$ be any vector. We have

$$\frac{k}{d} \boldsymbol{x}^T L_H \boldsymbol{x} = \sum_{i=1}^{n} \boldsymbol{x}^T \frac{k}{d} L_{H_i} \boldsymbol{x} \geq \sum_{i=1}^{n} (1-\epsilon) \boldsymbol{x}^T L_{K_i} \boldsymbol{x} = (1-\epsilon) \sum_{i=1}^{n} \boldsymbol{x}^T L_{K_i} \boldsymbol{x} = (1-\epsilon) \boldsymbol{x}^T (G \textcircled{L} K) \boldsymbol{x}.$$

$\square$

So, the spectral ratio of $G \textcircled{L} K$ is a little less than half that of G. But, the degree of $G \textcircled{L} K$ is $2k$, which we will arrange to be much less than the degree of G, d.

14.6 Squaring the graph

We can improve the spectral ratio of a graph by squaring it, at the cost of increasing its degree. Given a graph G, we define the graph G^2 to be the graph in which vertices u and v are connected if they are at distance 2 in G. Formally, G^2 should be a weighted graph in which the weight of an edge is the number of such paths. We may form the adjacency matrix of G^2 from the adjacency matrix of G. Let A be the adjacency matrix of G. Then $A^2(u,v)$ is the number of paths of length 2 between u and v in G, and $A^2(v,v)$ is always d. So,

$$A_{G^2} = A_G^2 - dI_n.$$

In our construction, it is easy to ensure that the graph we square has no triangles, so all of the edges in its square will have weight 1.

Lemma 14.6.1. *Let G be a d-regular graph whose second-smallest Laplacian eigenvalue is λ_2. Then, G^2 is a $d(d-1)$-regular graph whose second-smallest Laplacian eigenvalue is*

$$2d\lambda_2 - \lambda_2^2,$$

and whose largest Laplacian eigenvalue is at most d^2.

Proof. First, let's handle the largest Laplacian eigenvalue. As A_G^2 is positive semi-definite, the smallest eigenvalue of A_{G^2} is at least $-d$, and so the largest Laplacian eigenvalue of G^2 is at most

$$d(d-1) - d = d^2.$$

As for the other eigenvalues, we find that

$$\begin{aligned}
&\lambda_i \text{ is an eigenvalue of } L_G \implies \\
&d - \lambda_i \text{ is an eigenvalue of } A_G \implies \\
&(d-\lambda_i)^2 - d \text{ is an eigenvalue of } A_{G^2} \implies \\
&d(d-1) - (d-\lambda_i)^2 + d \text{ is an eigenvalue of } L_{G^2},
\end{aligned}$$

and

$$d(d-1) - (d-\lambda_i)^2 + d = d^2 - (d-\lambda_i)^2 = 2d\lambda_i - \lambda_i^2.$$

□

If the spectral ratio of G is small, then the spectral ratio of G^2 will be approximately twice the spectral ratio of G.

14.7 The whole construction

We begin with any small d-regular expander graph G_0. And let β be its spectral ratio. We will assume that β is small, but greater than 0, say 1/20. Of course, it does not hurt to start with a graph of larger spectral ratio.

We then construct $G_0 \textcircled{L} K$. The degree of this graph is $2k$, and its spectral ratio is a little less than $\beta/2$. So, we square the resulting graph, to obtain

$$(G_0 \textcircled{L} K)^2.$$

It has degree approximately $4k^2$, and spectral ratio slightly less than β. But, for induction, we need it to be more than β. So, we square one more time, to get a spectral ratio a little less than 2β. We now set

$$G_1 = \left((G_0 \textcircled{L} K)^2\right)^2.$$

As G_1 is a square, its largest Laplacian eigenvalue is extremely close to its degree. The graph G_1 is at least as good an approximation of a complete graph as G_0, and it has degree approximately $16k^4$. In general, we set

$$G_{i+1} = \left((G_i \textcircled{L} K)^2\right)^2.$$

To make the inductive construction work, we need for K to be a graph of degree k whose number of vertices equals the degree of G. This is approximately $16k^4$, and is exactly

$$(2k(2k-1))^2 - 2k(2k-1).$$

Last lecture we learned how to find such graphs that are very good approximations of complete graphs.

The graphs G_i that we construct are not all that good approximations of the complete graph: we just know that the ratio of the second-smallest Laplacian eigenvalue to the degree is bounded away from zero by β. To improve their spectral ratio, we can just square them a few times.

14.8 Better Constructions

There is a better construction technique, called the Zig-Zag product [RVW02]. The Zig-Zag construction is a little trickier to understand, but it achieves better expansion. I chose to present the line-graph based construction because its analysis is very closely related to an analysis of the Zig-Zag product.

References

[RVW02] Omer Reingold, Salil Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. *Annals of Mathematics*, 155(1):157–187, 2002.