Matrix Tree Theorems

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1 Counting Trees

The Laplacian of a graph can be used to easily obtain a wealth of information about its spanning trees. To talk about this, we will need to recall the *elementary symmetric functions*: let $e_k(A)$ denote the sum of $k-$wise products of eigenvalues of A, noting that $e_k(A) = \det(A)$ for a $k \times k$ matrix and $e_k(A) = 0$ for $k > \text{rank}(A)$. Observe that for a connected graph, $e_{n-1}(L_G)$ is simply the product of the nonzero eigenvalues. We now have the following celebrated theorem of Kirchoff.

Theorem 1. *Let $\mathbf{N}_G$ denote the number of spanning trees of G. Then*

$$e_{n-1}(L_G) = n \cdot \mathbf{N}_G.$$

The base case of this theorem is the following. We will use the notation $A_{R,C}$ to denote the submatrix of A with rows in R and columns in C.

Lemma 2. *Suppose L is the Laplacian of a tree T on n vertices. Then*

$$e_{n-1}(L) = n.$$

Proof. Orient the edges of T arbitrarily. Let $L = B^T B$ where B is the edge-vertex incidence matrix and use Lemma 3 to obtain

$$\begin{aligned} e_{n-1}(L) &= e_{n-1}(B^T B) \\ &= \sum_{x \in V} e_{n-1}(B^T_{\cdot,\bar{x}} B_{\cdot,\bar{x}}) \end{aligned}$$

We will show that each of the terms in the sum is 1.

Fix x. Take any vertex $y \neq x$ and consider the unique oriented path $P \subset T$ from x to y. Then it is easy to see that

$$\chi_y - \chi_x = \sum_{e \in T} z_y(e) b_e$$

where

$$z_y(e) = \begin{cases} +1 & \text{if } e \in P \text{ with the same orientation} \\ -1 & \text{if } e \in P \text{ with the opposite orientation} \\ 0 & \text{if } e \notin P \end{cases}$$

for edges $e \in T$. Writing this fact in matrix notation, we have

$$\chi_y - \chi_x = B^T z_y$$

which upon ignoring the row indexed by x becomes

$$\chi_y = B^T_{\cdot,\bar{x}} z_y.$$

Thus by expressing the canonical vectors $\{\chi_y\}_{y \neq x}$ in terms of the columns of $B^T_{\cdot,\bar{x}}$, we have actually shown that

$$B^T_{\cdot,\bar{x}} Z = I_{n-1}$$

where Z has columns z_y. But now both Z and $B^T_{\cdot,\bar{x}}$ have integer entries and

$$\det(I_{n-1}) = \det(ZB_{\cdot,\bar{x}}) = \det(Z)\det(B_{\cdot,\bar{x}}) = 1$$

so we must have $\det(B_{\cdot,\bar{x}}) = \pm 1$ and consequently

$$\det(B_{\cdot,\bar{x}} B^T_{\cdot,\bar{x}}) = 1,$$

as desired. ◻

The following lemma gives the elementary symmetric functions of a matrix in terms of those of its principal submatrices.

Lemma 3. *If A is an $n \times n$ matrix then*

$$e_k(A) = \sum_{S \subset [n], |S|=k} \det(A_{S,S}).$$

Proof. Evaluate $\det(xI + A)$ using the formula

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \dots a_{m\sigma(m)}$$

where σ ranges over permutations. It is easy to see that the only terms that contain x^{n-k} come from permutations that are the identity on some $n-k$ indices (since the only terms containing x lie on the diagonal) and any permutation on the remaining k indices. Since the fixed indices do not change the signs of permutations, the coefficient of x^{n-k} is just

$$\sum_{|S|=k} \sum_{\sigma : k \to k} \operatorname{sgn}(\sigma) A_{S,S}(1, \sigma(1)) \dots A_{S,S}(k, \sigma(k))$$

which is just the sum over all principal $k \times k$ determinants. On the other hand, the coefficient of x^{n-k} is also e_k, since det is unitarily invariant. ◻

We can now use the above lemma to "count" the trees in an arbitrary G.

Proof of Theorem 1. We compute

$$\begin{aligned}
e_{n-1}(L) &= e_{n-1}(B^T B) \\
&= e_{n-1}(BB^T) && \text{since } BB^T \text{ and } B^T B \text{ are similar} \\
&= \sum_{T\subset E, |T|=n-1} e_{n-1}(B_{T,\cdot} B_{T,\cdot}^T) && \text{by Lemma 3} \\
&= \sum_{T\subset E, |T|=n-1} e_{n-1}(L_T) \\
&= \sum_{T\subset E, |T|=n-1, \text{connected}} e_{n-1}(L_T) && \text{since disconnected } T \text{ have } \operatorname{rank}(L_T) < n-1 \\
&= \sum_{T\subset E, |T|=n-1, \text{connected}} n && \text{by Lemma 2}
\end{aligned}$$

as desired. □

2 Effective Resistances

Identify the graph G with an electrical circuit in which each edge corresponds to a unit resistor. We will use the following notation to describe electrical flows on G: for a vector $\mathbf{i}_{\text{ext}}(u)$ of currents injected at the vertices, let $\mathbf{i}(e)$ be the currents induced in the edges (in the direction of orientation) and $\mathbf{v}(u)$ the potentials induced at the vertices. By Kirchoff's current law, the sum of the currents entering a vertex is equal to the amount injected:

$$B^T \mathbf{i} = \mathbf{i}_{\text{ext}}.$$

By Ohm's law, the current flow in an edge is equal to the potential difference across its ends:

$$\mathbf{i} = B\mathbf{v},$$

Combining these two facts, we obtain

$$\mathbf{i}_{\text{ext}} = B^T(B\mathbf{v}) = L\mathbf{v}.$$

If $\mathbf{i}_{\text{ext}} \perp \mathbf{1} = \ker(L)$, then we can write

$$\mathbf{v} = L^+ \mathbf{i}_{\text{ext}}$$

where L^+ is the pseudoinverse.

Recall that the *effective resistance* between two vertices u and v is defined as the potential difference induced between them when a unit current is injected at one and extracted at the other. We will derive an algebraic expression for the effective resistance in terms of L^+. To inject and extract a unit current across the endpoints of an edge $e = (u, v)$, we set $\mathbf{i}_{\text{ext}} = b_e^T = (\chi_v - \chi_u)$, which is clearly orthogonal to $\mathbf{1}$. The potentials induced by $\mathbf{i}_{\text{ext}}$ at the vertices are given by $\mathbf{v} = L^+ b_e^T$; to measure the potential difference across $e = (u, v)$, we simply multiply by b_e on the left:

$$\mathbf{v}(v) - \mathbf{v}(u) = (\chi_v - \chi_u)^T \mathbf{v} = b_e L^+ b_e^T.$$

It follows that the effective resistance across e is given by $b_e L^+ b_e^T$ and that the matrix $\Pi = BL^+B^T$ has as its diagonal entries $\Pi(e, e) = \mathsf{R}_{\mathsf{eff}}(e)$.

We are now in a position to state the second theorem.

Theorem 4. *If T is a spanning tree of G chosen uniformly at random, then for every edge $e \in G$:*

$$\mathsf{R_{eff}}(e) = \mathbb{P}_T[e \in T].$$

Proof. It is easy to verify that $\Pi = \Pi^T$ and $\Pi^T \Pi = \Pi$, so that $\mathsf{R_{eff}}(e) = \Pi(e,e) = \|\Pi_e\|^2$ where Π_e denotes the e^{th} column of Π. It follows that Π is a projection matrix with exactly $n-1$ eigenvalues, all equal to one.

Fix an edge e. Taking a sum over all $(n-1) \times (n-1)$ submatrices of Π containing e, we find that

$$\begin{aligned}
\sum_{T \subset E, T \ni e} e_{n-1}(\Pi_{T,\cdot} \Pi_{T,\cdot}^T) &= \sum_{T \subset E, T \ni e} e_{n-1}(B_{T,\cdot} L_G^+ B_{T,\cdot}^T) \\
&= \sum_{T \subset E, T \ni e} e_{n-1}(L_G^+ L_T) \\
&= \sum_{T \subset E, T \ni e} e_{n-1}(L_G^+) e_{n-1}(L_T) && \text{since } \mathrm{im}(L_G) = \mathrm{im}(L_T) \\
&= \sum_{T \subset E, T \ni e} \frac{e_{n-1}(L_T)}{e_{n-1}(L_G)} && \text{since } e_{n-1}(L_G)^{-1} = e_{n-1}(L_G^+) \\
&= \frac{|\{\text{spanning tree } T : T \ni e\}|}{\mathbf{N}_G} && \text{by Theorem 1 and Lemma 2} \\
&= \mathbb{P}_T[e \in T].
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
&\sum_{T \subset E, T \ni e} e_{n-1}(\Pi_{T,\cdot} \Pi_{T,\cdot}^T) \\
&= \sum_{T \subset E, T \ni e} \|\Pi_e\|^2 e_{n-2}(\Gamma_{\perp e} \Pi_{T,\cdot} \Pi_{T,\cdot}^T \Gamma_{\perp e}^T) && \text{where } \Gamma_{\perp e} \text{ is the projection orthogonal to } \Pi_e \\
&= \|\Pi_e\|^2 \sum_{T \subset E \setminus \{e\}, |T| = n-2} e_{n-2}(\Gamma_{\perp e} \Pi_{T \subset E,\cdot} \Pi_{T,\cdot}^T \Gamma_{\perp e}^T) \\
&= \|\Pi_e\|^2 e_{n-2}(\Gamma_{\perp e} \Pi \Pi^T \Gamma_{\perp e}^T) && \text{by Lemma 3} \\
&= \|\Pi_e\|^2 \cdot 1 && \text{since } \Gamma_{\perp e} \Pi \text{ has } n-2 \text{ nonzero eigenvalues equal to 1} \\
&= \mathsf{R_{eff}}(e),
\end{aligned}$$

as desired. $\square$