Spectral Graph Theory Lecture 20

Eigenvalues of Random Graphs

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20.1 Introduction

In this lecture, we consider a random graph on n vertices in which each edge is chosen to be in the graph with probability one-half, independently of course. We will show that the eigenvalues of the adjacency matrix of such a graph are tightly concentrated. Curiously, the adjacency matrix eigenvalues are much more tightly concentrated than the Laplacian matrix eigenvalues.

The adjacency matrix of such a random graph may be described by choosing the values of $A(i, j)$ to be zero with probability $1/2$ and 1 with probability $1/2$, subject to $A(i, j) = A(j, i)$. Of course, we fix $A(i, i) = 0$ for all i. The expectation of every off-diagonal entry of the matrix is $1/2$. Let M denote this expected matrix, and observe that

$$M = \frac{1}{2} A_{K_n} = \frac{1}{2}(J_n - I_n),$$

where A_{K_n} is the adjacency matrix of the compelete graph on n vertices, J_n is the all-1s matrix and I_n is of course the identity. From this formula, we see that M has one eigenvalue of $(n - 1)/2$ and $n - 1$ eigenvalues of $-1/2$. We will show that the eigenvalues of A are very close to this. In particular, we will prove that

$$\|A - M\| \leq (5/3)\sqrt{n},$$

with exponentially high probability. If this holds, you may apply problem 1 from problem set 2 to show that A has one eigenvalue within $(5/3)\sqrt{n}$ of $(n-1)/2$ and $n-1$ eigenvalues within $(5/3)\sqrt{n}$ of $-1/2$.

So, we will really focus on bounding the norm of $A - M$. As $A - M$ is a symmetric matrix, we have

$$\|A - M\| = \max_i |\lambda_i(A - M)| = \max_{\boldsymbol{x}} \left| \frac{\boldsymbol{x}^T A \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} \right|.$$

Our analysis will focus on this last term.

Set

$$R = A - M,$$

and let $r_{i,j} = R(i, j)$, for $i < j$. Each $r_{i,j}$ is a random variable that is independently and uniformly distributed in $\pm 1/2$.

20.2 One Rayleigh Quotient

To begin, we fix *any* unit vector $\boldsymbol{x}$, and consider

$$\boldsymbol{x}^T R \boldsymbol{x} = \sum_{i<j} 2 r_{i,j} \boldsymbol{x}(i) \boldsymbol{x}(j).$$

This is a sum of independent random variables, and so may be proved to be tighly concentrated around its expectation, which in this case is zero. There are many types of concentration bounds, with the most popular being the Chernoff and Hoeffding bounds. In this case we will apply Hoeffding's inequality.

Theorem 20.2.1 (Hoeffding's Inequality)**.** *Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ be real numbers and let $X_1, \ldots, X_m$ be independent random variables such that X_i takes values between a_i and b_i. Let $\mu = \mathbf{E}\left[\sum_i X_i\right]$. Then, for every $t > 0$,*

$$Pr\left[\sum X_i \geq \mu + t\right] \leq \exp\left(-\frac{2t^2}{\sum_i (b_i - a_i)^2}\right).$$

To apply this theorem, we view

$$X_{i,j} = 2 r_{i,j} \boldsymbol{x}(i) \boldsymbol{x}(j)$$

as our random variables. As $r_{i,j}$ takes values in $\pm 1/2$, we can set

$$a_{i,j} = -\boldsymbol{x}(i)\boldsymbol{x}(j) \quad \text{and} \quad b_{i,j} = \boldsymbol{x}(i)\boldsymbol{x}(j).$$

We then compute

$$\sum_{i<j} (b_i - a_i)^2 = \sum_{i<j} 4 \boldsymbol{x}(i)^2 \boldsymbol{x}(j)^2 = 2 \sum_{i \neq j} \boldsymbol{x}(i)^2 \boldsymbol{x}(j)^2 \leq 2 \left(\sum_i \boldsymbol{x}(i)^2\right)\left(\sum_i \boldsymbol{x}(j)^2\right) = 2,$$

as $\boldsymbol{x}$ is a unit vector.

We thereby obtain the following bound on $\boldsymbol{x}^T R \boldsymbol{x}$.

Lemma 20.2.2. *For every unit vector $\boldsymbol{x}$,*

$$Pr_R\left[\left|\boldsymbol{x}^T R \boldsymbol{x}\right| \geq t\right] \leq 2e^{-t^2}.$$

Proof. The expectation of $\boldsymbol{x}^T R \boldsymbol{x}$ is 0. The preceeding argument tells us that

$$\begin{aligned}
\Pr\left[\left|\boldsymbol{x}^T R \boldsymbol{x}\right| \geq t\right] &\leq \Pr\left[\boldsymbol{x}^T R \boldsymbol{x} \geq t\right] + \Pr\left[\boldsymbol{x}^T R \boldsymbol{x} \leq -t\right] \\
&\leq \Pr\left[\boldsymbol{x}^T R \boldsymbol{x} \geq t\right] + \Pr\left[\boldsymbol{x}^T (-R) \boldsymbol{x} \geq t\right] \\
&\leq 2e^{-t^2},
\end{aligned}$$

where we have exploited the fact that R and $-R$ are identically distributed. $\square$

20.3 Vectors near $\boldsymbol{v}_1$

You might be wondering what good the previous argument will do us. We have shown that it is unlikely that the Rayleigh quotient of any given $\boldsymbol{x}$ is large. But, we have to reason about all $\boldsymbol{x}$ of unit norm.

Lemma 20.3.1. *Let R be a symmetric matrix and let $\boldsymbol{v}$ be a unit eigenvector of R whose eigenvalue has absolute value $\|R\|$. If $\boldsymbol{x}$ is another unit vector such that*

$$\boldsymbol{v}^T \boldsymbol{x} \geq \sqrt{3}/2,$$

then

$$\boldsymbol{x}^T R \boldsymbol{x} \geq \frac{1}{2} \|R\| .$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of R and let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be a corresponding set of orthonormal eigenvectors. Assume without loss of generality that $\lambda_1 \geq |\lambda_n|$ and that $\boldsymbol{v} = \boldsymbol{v}_1$. Expand $\boldsymbol{x}$ in the eigenbasis as

$$\boldsymbol{x} = \sum_i c_i \boldsymbol{v}_i .$$

We know that $c_1 \geq \sqrt{3}/2$ and $\sum_i c_i^2 = 1$. This implies that

$$\boldsymbol{x}^T R \boldsymbol{x} = \sum_i c_i^2 \lambda_i \geq c_1^2 \lambda_1 - \sum_{i \geq 2} c_i^2 |\lambda_1| = \lambda_1 \left(c_1^2 - \sum_{i \geq 2} c_i^2 \right) = \lambda_1 (2c_1^2 - 1) \geq \lambda_1/2.$$

□

We will bound the probability that $\|R\|$ is large by taking Rayleigh quotients with random unit vectors. Let's examine the probability that a random unit vector $\boldsymbol{x}$ satisfies the conditions of Lemma 20.3.1.

Lemma 20.3.2. *Let $\boldsymbol{v}$ be an arbitrary unit vector, and let $\boldsymbol{x}$ be a random unit vector. Then,*

$$Pr \left[\boldsymbol{v}^T \boldsymbol{x} \geq \sqrt{3}/2 \right] \geq \frac{1}{\sqrt{\pi} n 2^{n-1}}$$

Proof. Let B^n denote the unit ball in $\mathbb{R}^n$, and let C denote the cap on the surface of B^n containing all vectors $\boldsymbol{x}$ such that

$$\boldsymbol{v}^T \boldsymbol{x} \geq \sqrt{3}/2.$$

We need to lower bound the ratio of the surface area of the cap C to the surface area of B^n.

Recall that the surface area of B^n is

$$\frac{n \pi^{n/2}}{\Gamma(\frac{n}{2} + 1)},$$

where I recall that for positive integers n

$$\Gamma(n) = (n-1)!,$$

and that $\Gamma(x)$ is an increasing function for real $x \geq 1$.

Now, consider the $(n-1)$-dimensional hypersphere whose boundary is the boundary of the cap C. As the cap C lies above this hypersphere, the $(n-1)$-dimensional volume of this hypersphere is a lower bound on the surface area of the cap C. Recall that the volume of a sphere in $\mathbb{R}^n$ of radius r is

$$r^n \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}.$$

In our case, the radius of the hypersphere is

$$r = \sin(\mathrm{acos}\sqrt{3}/2) = 1/2.$$

So, the ratio of the $(n-1)$-dimensional volume of the hypersphere to the surface area of B^n is at least

$$\frac{r^{n-1}\frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2}+1)}}{\frac{n\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}} = \frac{r^{n-1}}{\sqrt{\pi}n}\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n-1}{2}+1)} \geq \frac{1}{\sqrt{\pi}n2^{n-1}}.$$

□

20.4 The Probabilistic Argument

I'm going to do the following argument very slowly, because it is both very powerful and very subtle.

Theorem 20.4.1. *Let R be a symmetric matrix with zero diagonal and off-diagonal entries uniformly chose from $\pm 1/2$. Then,*

$$Pr\left[\|R\| \geq t\right] \leq \sqrt{\pi}n2^n e^{-t^2/4}.$$

Proof. Let R be a fixed symmetric matrix. By applying Lemma 20.3.2 to any eigenvector of R whose eigenvalue has maximal absolute value, we find

$$\mathrm{Pr}_{\boldsymbol{x}}\left[\left|\boldsymbol{x}^T R\boldsymbol{x}\right| \geq \frac{1}{2}\|R\|\right] \geq \frac{1}{\sqrt{\pi}n2^{n-1}}.$$

Thus, for a random R we find

$$\mathrm{Pr}_{R,\boldsymbol{x}}\left[\|R\| \geq t \text{ and } \left|\boldsymbol{x}^T R\boldsymbol{x}\right| \geq \frac{1}{2}\|R\|\right] \geq \mathrm{Pr}_R\left[\|R\| \geq t\right]\frac{1}{\sqrt{\pi}n2^{n-1}}.$$

On the other hand,

$$\begin{aligned}\mathrm{Pr}_{R,\boldsymbol{x}}\left[\|R\| \geq t \text{ and } \left|\boldsymbol{x}^T R\boldsymbol{x}\right| \geq \frac{1}{2}\|R\|\right] &\leq \mathrm{Pr}_{R,\boldsymbol{x}}\left[\|R\| \geq t \text{ and } \left|\boldsymbol{x}^T R\boldsymbol{x}\right| \geq t/2\right]\\ &\leq \mathrm{Pr}_{R,\boldsymbol{x}}\left[\left|\boldsymbol{x}^T R\boldsymbol{x}\right| \geq t/2\right]\\ &\leq \mathrm{Pr}_{\boldsymbol{x}}\left[\mathrm{Pr}_R\left[\left|\boldsymbol{x}^T R\boldsymbol{x}\right| \geq t/2\right]\right]\\ &\leq 2e^{-(t/2)^2},\end{aligned}$$

where the last inequality follows from Lemma 20.2.2.

Combining these inequalities, we obtain

$$\Pr_R [\|R\| \geq t] \frac{1}{\sqrt{\pi} n 2^{n-1}} \leq e^{-(t/2)^2},$$

which implies the claimed result. $\square$

The probability in Theorem 20.4.1 becomes small once $e^{t^2/4}$ exceeds $\sqrt{\pi} n 2^n$. As n grows large, this happens for

$$t > 2\sqrt{\ln 2}\sqrt{n} \sim (5/3)\sqrt{n}.$$

It is known that the norm of R is unlikely to be much more than $\sqrt{n}$. This is proved by Füredi and Komlós [FK81] and Vu [Vu07], using a very different technique. The idea behind these papers is to consider $\mathrm{Tr}\left(R^k\right)$ for a high power of k. They show that the expectation of this variable is unlikely to be large, and exploit the fact that

$$\|R\| \leq \left(\mathrm{Tr}\left(R^k\right)\right)^{1/k}.$$

References

[FK81] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.

[Vu07] Van Vu. Spectral norm of random matrices. *Combinatorica*, 27(6):721–736, 2007.