**Spectral Graph Theory** Lecture 23

Strongly Regular Graphs, part 1

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## 23.1 Introduction

In this and the next lecture, I will discuss strongly regular graphs. Strongly regular graphs are extremal in many ways. For example, their adjacency matrices have only three distinct eigenvalues. If you are going to understand spectral graph theory, you must have these in mind.

In many ways, strongly-regular graphs can be thought of as the high-degree analogs of expander graphs. However, they are much easier to construct.

Many times someone has asked me for a matrix of 0s and 1s that "looked random", and strongly regular graphs provided a resonable answer.

**Warning:** I will use the letters that are standard when discussing strongly regular graphs. So $\lambda$ and $\mu$ will not be eigenvalues in this lecture.

## 23.2 Definitions

Formally, a graph $G$ is strongly regular if

1. it is $k$-regular, for some integer $k$;
2. there exists an integer $\lambda$ such that for every pair of vertices $x$ and $y$ that are neighbors in $G$, there are $\lambda$ vertices $z$ that are neighbors of *both* $x$ and $y$;
3. there exists an integer $\mu$ such that for every pair of vertices $x$ and $y$ that are *not* neighbors in $G$, there are $\mu$ vertices $z$ that are neighbors of *both* $x$ and $y$.

These conditions are very strong, and it might not be obvious that there are any non-trivial graphs that satisfy these conditions. Of course, the complete graph and disjoint unions of complete graphs satisfy these conditions.

For the rest of this lecture, we will only consider strongly regular graphs that are connected and that are not the complete graph. I will now give you some examples.

## 23.3 The Pentagon

The simplest strongly-regular graph is the pentagon. It has parameters

$$n = 5, \quad k = 2, \quad \lambda = 0, \quad \mu = 1.$$

## 23.4 Lattice Graphs

For a positive integer $n$, the *lattice graph* $L_n$ is the graph with vertex set $\{1, \ldots n\}^2$ in which vertex $(a, b)$ is connected to vertex $(c, d)$ if $a = c$ or $b = d$. Thus, the vertices may be arranged at the points in an $n$-by-$n$ grid, with vertices being connected if they lie in the same row or column. Alternatively, you can understand this graph as the line graph of a bipartite complete graph between two sets of $n$ vertices.

It is routine to see that the parameters of this graph are:

$$k = 2(n-1), \quad \lambda = n-2, \quad \mu = 2.$$

## 23.5 Latin Square Graphs

A Latin Square is an $n$-by-$n$ grid, each entry of which is a number between 1 and $n$, such that no number appears twice in any row or column. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Let me remark that the number of different latin squares of size $n$ grows very quickly, at least as fast as $n!(n-1)!(n-2)! \ldots 2!$.

From such a latin square, we construct a Latin Square Graph. It will have $n^2$ nodes, one for each cell in the square. Two nodes are joined by an edge if

1. they are in the same row,
2. they are in the same column, or
3. they hold the same number.

So, such a graph has degree $k = 3(n-1)$. Any two nodes in the same row will both be neighbors with every other pair of nodes in their row. They will have two more common neighors: the nodes in their columns holding the other's number. So, they have $n$ common neighbors. The same obviously holds for columns, and is easy to see for nodes that have the same number. So, every pair of nodes that are neighbors have exactly $\lambda = n$ common neighbors.

On the other hand, consider two vertices that are not neighbors, say $(1,1)$ and $(2,2)$. They lie in different rows, lie in different columns, and hold different numbers. The vertex $(1,1)$ has two common neighbors of $(2,2)$ in its row: the vertex $(1,2)$ and the vertex holding the same number as $(2,2)$. Similarly, it has two common neighbors of $(2,2)$ in its column. Finally, we can find two more common neighbors of $(2,2)$ that are in different rows and columns by looking at the nodes that hold the same number as $(1,1)$, but which are in the same row or column as $(2,2)$. So, $\mu = 6$.

## 23.6 The Eigenvalues of Strongly Regular Graphs

We will consider the adjacency matrices of strongly regular graphs. Let $A$ be the adjacency matrix of a strongly regular graph with parameters $(k, \lambda, \mu)$. We already know that $A$ has an eigenvalue of $k$ with multiplicity 1. We will now show that $A$ has just two other eigenvalues.

To prove this, first observe that the $(u, v)$ entry of $A^2$ is the number of common neighbors of vertices $u$ and $v$. For $u = v$, this is just the degree of vertex $u$. We will use this fact to write $A^2$ as a linear combination of $A$, $I$ and $J$. To this end, observe that the adjacency matrix of the complement of $A$ (the graph with non-edges where $A$ has edges) is $J - I - A$. So,

$$A^2 = \lambda A + \mu(J - I - A) + kI = (\lambda - \mu)A + \mu J + (k - \mu)I.$$

For every vector $\boldsymbol{v}$ orthogonal to $\mathbf{1}$,

$$A^2\boldsymbol{v} = (\lambda - \mu)A\boldsymbol{v} + (k - \mu)\boldsymbol{v}.$$

So, every eigenvalue $\theta$ of $A$ other than $k$ satisfies

$$\theta^2 = (\lambda - \mu)\theta + k - \mu.$$

The eigenvalues of $A$ other than $k$ are those $\theta$ that satisfy this quadratic equation, and so are given by

$$\frac{\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}.$$

These eigenvalues are always denoted $r$ and $s$, with $r > s$. By convention, the multiplicty of the eigenvalue $r$ is always denoted $f$, and the multiplicty of $s$ is always denoted $g$.

For example, for the pentagon we have

$$r = \frac{\sqrt{5} - 1}{2}, \quad s = -\frac{\sqrt{5} + 1}{2}.$$

For the lattice graph $L_n$, we have

$$r = n - 2, \quad s = -2.$$

For the latin square graphs of order $n$, we have

$$r = n - 3, \quad s = -3.$$

## 23.7 Regular graphs with three eigenvalues

We will now show that every regular connected graph with at most 3 eigenvalues must be a strongly regular graph. Let $G$ be $k$-regular, and let its eigenvalues other than $k$ be $r$ and $s$. As $G$ is connected, its adjacency eigenvalue $k$ has multiplicty 1.

Then, for every vector orthogonal to $\mathbf{1}$, we have

$$(A - rI)(A - sI)\boldsymbol{v} = 0.$$

Thus, for some $\beta$,

$$(A - rI)(A - sI) = \beta J,$$

which gives

$$\begin{aligned} A^2 - (r+s)A + rsI = \beta J \quad &\Longrightarrow \\ A^2 &= (r+s)A - rsI + \beta J \\ &= (r+s+\beta)A + \beta(J - A - I) + (rs+\beta)I. \end{aligned}$$

So, the number of common neighbors of two nodes just depends on whether or not they are neighbors, which implies that $A$ is strongly regular.

## 23.8 Integrality of the eigenvalues

We will now see that, unless $f = g$, both $r$ and $s$ must be integers. We do this by observing a few identities that they both must satisfy. First, from the quadratic equation above, we know that

$$r + s = \lambda - \mu \tag{23.1}$$

and

$$rs = \mu - k. \tag{23.2}$$

As the trace of an adjacency matrix is zero, and is also the sum of the eigenvalues times their multiplicites, we know

$$k + fr + gs = 0. \tag{23.3}$$

So, it must be the case that $s < 0$. Equation 23.1 then gives $r > 0$.

If $f \neq g$, then equations (23.3) and (23.1) provide independent constraints on $r$ and $s$, and so together they determine $r$ and $s$. As the coefficients in both equations are integers, they tell us that both $r$ and $s$ are rational numbers. From this, and the fact that $r$ and $s$ are the roots of a quadratic equation with integral coefficients, we may conclude that $r$ and $s$ are in fact integers. Let me remind you as to why.

**Lemma 23.8.1.** *If $\theta$ is a rational number that satisfies*

$$\theta^2 + b\theta + c = 0,$$

*where $b$ and $c$ are integers, then $\theta$ must be an integer.*

*Proof.* Write $\theta = x/y$, where the greatest common divisor of $x$ and $y$ is 1. We then have

$$(x/y)^2 + b(x/y) + c = 0,$$

which implies

$$x^2 + bxy + cy^2 = 0,$$

which implies that $y$ divides $x^2$. As we have assumed the greatest common divisor of $x$ and $y$ is 1, this implies $y = 1$. $\square$

## 23.9 The Eigenspaces of Strongly Regular Graphs

It is natural to ask what the eigenspaces can tell us about a strongly regular graph. But, we will find that they don't tell us anything we don't already know.

Let $\boldsymbol{u}_1, \dots \boldsymbol{u}_f$ be an orthonormal set of eigenvectors of the eigenvalue $r$, and let $U$ be the matrix containing these vectors as columns. Recall that $U$ is only determined up to an orthnormal transformation. That is, we could equally take $UQ$ for any $f$-by-$f$ orthnormal matrix $Q$.

To the $i$th vertex, we associate the vector

$$\boldsymbol{x}_i \stackrel{\text{def}}{=} (\boldsymbol{u}_1(i), \dots, \boldsymbol{u}_f(i)).$$

While the vectors $U$ are determined only up to orthogonal transformations, these transformations don't effect the geometry of these vectors. For example, for vertices $i$ and $j$, the distance between $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ is

$$\left\| \boldsymbol{x}_i - \boldsymbol{x}_j \right\|,$$

and

$$\left\| \boldsymbol{x}_i - \boldsymbol{x}_j \right\|^2 = \left\| \boldsymbol{x}_i \right\|^2 + \left\| x\boldsymbol{x}_j \right\|^2 - 2\boldsymbol{x}_i \boldsymbol{x}_j^T.$$

On the other hand,

$$\left\| \boldsymbol{x}_i Q - \boldsymbol{x}_j Q \right\|^2 = \left\| \boldsymbol{x}_i Q \right\|^2 + \left\| x\boldsymbol{x}_j Q \right\|^2 - 2(\boldsymbol{x}_i Q)(\boldsymbol{x}_j Q)^T = \left\| \boldsymbol{x}_i Q \right\|^2 + \left\| x\boldsymbol{x}_j Q \right\|^2 - 2\boldsymbol{x}_i Q Q^T \boldsymbol{x}_j^T = \left\| \boldsymbol{x}_i \right\|^2 + \left\| x\boldsymbol{x}_j \right\|^2 - 2\boldsymbol{x}_i \boldsymbol{x}$$

In fact, all the geometrical information about the vectors $\boldsymbol{x}_i$ is captured by their Gram matrix, whose $(i,j)$ entry is $\boldsymbol{x}_i \boldsymbol{x}_j^T$. This matrix is also given by

$$UU^T.$$

Let $W$ be the analogous matrix for the eigenvalue $s$. We then have

$$A = rUU^T + sWW^T + k\frac{1}{n}J.$$

As each of the matrices $UU^T$, $WW^T$ and $\frac{1}{n}J$ are projections (having all eigenvalues 0 or 1), and are mutually orthogonal, we also have

$$A^2 = r^2 UU^T + s^2 WW^T + k^2 \frac{1}{n}J.$$

Consider the polynomial

$$P(X) = \frac{(X-s)(X-k)}{(r-s)(r-k)}.$$

We have

$$P(X) = \begin{cases} 1 & \text{if } X = r \\ 0 & \text{if } X = s \text{, and} \\ 0 & \text{if } X = k. \end{cases}$$

So,

$$P(A) = P(r)UU^T + P(s)WW^T + P(k)\frac{1}{n}J = UU^T.$$

That is, the Gram matrix of the point set $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is a linear combination of the identity, $A$ and $A^2$. So, the distance between any pair of points in this set just depends on whether or not the corresponding vertices are neighbors in $G$.

In particular, this means that the point set $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is a *two-distance point set*: a set of points such that there are only two different distances between them. Next lecture, we will use this fact to prove a lower bound on the dimensions $f$ and $g$.

## 23.10 Triangular Graphs

For a positive integer $n$, the *triangular graph* $T_n$ may be defined to be the line graph of the complete graph on $n$ vertices. To be more concrete, its vertices are the subsets of size 2 of $\{1, \ldots, n\}$. Two of these sets are connected by an edge if their intersection has size 1.

You are probably familiar with some triangular graphs. $T_3$ is the triangle, $T_4$ is the skeleton of the octahedron, and $T_5$ is the complement of the Petersen graph.

Let's verify that these are strongly-regular, and compute their parameters. As the construction is competely symmetric, we may begin by considering any vertex, say the one labeled by the set $\{1, 2\}$. Every vertex labeled by a set of form $\{1, i\}$ or $\{2, i\}$, for $i \geq 3$, will be connected to this set. So, this vertex, and every vertex, has degree $2(n-2)$.

For any neighbor of $\{1, 2\}$, say $\{1, 3\}$, every other vertex of from $\{1, i\}$ for $i \geq 4$ will be a neighbor of both of these, as will the set $\{2, 3\}$. Carrying this out in general, we find that $\lambda = (n-3)+1 = n-2$.

Finally, any non-neighbor of $\{1, 2\}$, say $\{3, 4\}$, will have 4 common neighbors with $\{1, 2\}$:

$$\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}.$$

So, $\mu = 4$.

## 23.11 Paley Graphs

Probably next lecture.