Spectral Graph Theory Lecture 26

Planar Graphs 2, the Colin de Verdière Number

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26.1 Introduction

In this lecture, I will introduce the Colin de Verdière number of a graph, and sketch the proof that it is three for planar graphs. Along the way, I will recall two important facts about planar graphs:

1. Three-connected planar graphs are the skeletons of three-dimensional convex polytopes.
2. Planar graphs are the graphs that do not have K_5 or $K_{3,3}$ minors.

26.2 Colin de Verdière invariant

The Colin de Verdière graph paramater essentially measures the maximum multiplicity of the second eigenvalue of a generalized Laplacian matrix of the graph. It is less than or equal to three precisely for planar graphs.

We say that M is a *Generalized Laplacian Matrix* of a graph $G = (V, E)$ if M can be expressed as $M = L + D$ where L is a the Laplacian matrix of a weighted version of G and D is an arbitrary diagonal matrix. That is, we impose the restrictions:

$$\begin{aligned} &M(i,j) < 0 \quad \text{if } (i,j) \in E \\ &M(i,j) = 0 \quad \text{if } (i,j) \notin E \text{ and } i \neq j \\ &M(i,i) \text{ is arbitrary.} \end{aligned}$$

The Colin de Verdière graph paramater, which we denote $\mathsf{cdv}(G)$ is the maximum multiplicty of the second-smallest eigenvalue of a Generalized Laplacian Matrix M of G satisfying the following condition, known as the Strong Arnold Property.

> For every non-zero n-by-n matrix X such that $X(i,j) = 0$ for $i = j$ and $(i,j) \in E$, $MX \neq \mathbf{0}$.

That later restriction will be unnecessary for the results we will prove in this lecture.

Colin de Verdière [dV90] proved that $\mathsf{cdv}(G)$ is at most 2 if and only if the graph G is outerplanar. That is, it is a planar graph in which every vertex lies on one face. He also proved that it is at most 3 if and only if G is planar. Lovàsz and Schrijver [LS98] proved that it is at most 4 if and only if the graph is linkless embeddable.

In this lecture, I will sketch proofs from two parts of this work:

1. If G is a three-connected planar graph, then $\mathsf{cdv}(G) \geq 3$.
2. If G is a three-connected planar graph, then $\mathsf{cdv}(G) \leq 3$.

The first requires the construction of a matrix, which we do using the representation of the graph as a convex polytope. The second requires a proof that no Generalized Laplacian Matrix of the graph has a second eigenvalue of high multiplicity. We prove this by using graph minors.

26.3 Polytopes and Planar Graphs

Let me begin by giving two definitions of convex polytopes: as the convex hull of a set of points and as the intersection of half-spaces.

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^d$ (think $d = 3$). Then, the *convex hull* of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is the set of points

$$\left\{ \sum_i a_i \boldsymbol{x}_i : \sum a_i = 1 \text{ and all } a_i \geq 0 \right\}.$$

Every convex polytope is the convex hull of its extreme vertices.

A convex polytope can also be defined by its faces. For example, given vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_l$, the set of points

$$\left\{ \boldsymbol{x} : \boldsymbol{y}_i^T \boldsymbol{x} \leq 1, \text{ for all } i \right\}$$

is a convex polytope. Moreover, every convex polytope containing the origin in its interior can be described in this way. Each vector $\boldsymbol{y}_i$ defines a *face* of the polytope consisting of those points $\boldsymbol{x}$ in the polytope such that $\boldsymbol{y}_i^T \boldsymbol{x} = 1$.

The vertices of a convex polytope are those points $\boldsymbol{x}$ in the polytope that cannot be expressed non-trivially as a convex combination of any points other than themselves. The edges (or 1-faces) of a convex polytope are the line segments on the boundary of the polytope that go between two vertices of the polytope and such that every point on the edge cannot be expressed non-trivially as the convex hull of any vertices other than these two.

Theorem 26.3.1 (Steinitz's Theorem)**.** *For every three-connected planar graph $G = (V, E)$, there exists a set of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^3$ such that the line segment from $\boldsymbol{x}_i$ to $\boldsymbol{x}_j$ is an edge of the convex hull of the vectors if and only if $(i, j) \in E$.*

That is, every planar graph may be represented by the edges of a three-dimensional convex polytope. We will use this representation to construct a Generalized Laplacian Matrix M whose second-smallest eigenvalue has multiplicity 3.

26.4 The Colin de Verdière Matrix

Let $G = (V, E)$ be a planar graph, and let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^3$ be the vectors given by Steinitz's Theorem. For $1 \leq i \leq 3$, let $\boldsymbol{v}_i \in \mathbb{R}^n$ be the vector given by

$$\boldsymbol{v}_i(j) = \boldsymbol{x}_j(i).$$

So, the vector $\boldsymbol{v}_i$ contains the ith coordinate of each vector $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.

We will now see how to construct a generalized Laplacian matrix M having the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ in its nullspace. One can also show that the matrix M has precisely one negative eigenvalue. But, we won't have time to do that in this lecture. You can find the details in [Lov01].

Our construction will exploit the *vector cross product*. Recall that for two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^3$ that it is possible to define a vector $\boldsymbol{x} \times \boldsymbol{y}$ that is orthogonal to both $\boldsymbol{x}$ and $\boldsymbol{y}$, and whose length is the area of the parallelogram with sizes $\boldsymbol{x}$ and $\boldsymbol{y}$. This determines the cross product up to sign. You should recall that the sign is determined by an ordering of the basis of $\mathbb{R}^3$, or by the *right hand rule*. Also recall that

$$\begin{aligned} \boldsymbol{x} \times \boldsymbol{y} &= -\boldsymbol{y} \times \boldsymbol{x}, \\ (\boldsymbol{x}_1 + \boldsymbol{x}_2) \times \boldsymbol{y} &= \boldsymbol{x}_1 \times \boldsymbol{y} + \boldsymbol{x}_2 \times \boldsymbol{y}, \qquad \text{and} \\ \boldsymbol{x} \times \boldsymbol{y} &= 0 \text{ if and only if } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are parallel.} \end{aligned}$$

We will now specify the entries $M(i, j)$ for $(i, j) \in E$. An edge (i, j) is on the boundary of two faces of the polytope. Let's say that the vectors defining these faces are $\boldsymbol{y}_a$ and $\boldsymbol{y}_b$. So,

$$\boldsymbol{y}_a^T \boldsymbol{x}_i = \boldsymbol{y}_a^T \boldsymbol{x}_j = \boldsymbol{y}_b^T \boldsymbol{x}_i = \boldsymbol{y}_b^T \boldsymbol{x}_j = 1.$$

So,

$$(\boldsymbol{y}_a - \boldsymbol{y}_b)^T \boldsymbol{x}_i = (\boldsymbol{y}_a - \boldsymbol{y}_b)^T \boldsymbol{x}_j = 0.$$

This implies that $\boldsymbol{y}_a - \boldsymbol{y}_b$ is parallel to $\boldsymbol{x}_i \times \boldsymbol{x}_j$.

Assume $\boldsymbol{y}_a$ comes before $\boldsymbol{y}_b$ in the clockwise order about vertex $\boldsymbol{x}_i$. So, $\boldsymbol{y}_b - \boldsymbol{y}_a$ points the same direction as $\boldsymbol{x}_i \times \boldsymbol{x}_j$. Set $M(i, j)$ so that

$$M(i, j)\boldsymbol{x}_i \times \boldsymbol{x}_j = \boldsymbol{y}_a - \boldsymbol{y}_b$$

and $M(i, j) < 0$.

I will now show that we can choose the diagonal entries $M(i, i)$ so that the coordinate vectors are in the nullspace of M. First, set

$$\hat{\boldsymbol{x}}_i = \sum_{j \sim i} M(i, j)\boldsymbol{x}_j.$$

We will show that $\hat{\boldsymbol{x}}_i$ is parallel to $\boldsymbol{x}_i$ by observing that $\hat{\boldsymbol{x}}_i \times \boldsymbol{x}_i = 0$. We compute

$$\boldsymbol{x}_i \times \hat{\boldsymbol{x}}_i = \boldsymbol{x}_i \times \sum_{j \sim i} M(i, j)\boldsymbol{x}_j = \sum_{j \sim i} M(i, j)\boldsymbol{x}_i \times \boldsymbol{x}_j.$$

This sum counts the difference $\boldsymbol{y}_b - \boldsymbol{y}_a$ between each adjacent pair of faces that touch $\boldsymbol{x}_i$. By going around $\boldsymbol{x}_i$ in counter-clockwise order, we see that each of these vectors occurs once positively and once negatively in the sum, so the sum is zero.

Thus, $\boldsymbol{x}_i$ and $\hat{\boldsymbol{x}}_i$ are parallel, and we may set $M(i,i)$ so that

$$M(i,i)\boldsymbol{x}_i + \hat{\boldsymbol{x}}_i = \mathbf{0}.$$

This implies that the coordinate vectors are in the nullspace of M, as

$$\left(M \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_n \end{pmatrix} \right)_i = M(i,i)\boldsymbol{x}_i + \sum_{j \sim i} M(i,j)\boldsymbol{x}_j = M(i,i)\boldsymbol{x}_i + \hat{\boldsymbol{x}}_i.$$

One can also show that the matrix M has precisely one negative eigenvalue, so the multiplicity of its second-smallest eigenvalue is 3.

26.5 Minors of Planar Graphs

I will now show you that $\mathsf{cdv}(G) \leq 3$ for every 3-connected planar graph G. To begin, I mention one other characterization of planar graphs.

First, observe that if G is a planar graph, it remains planar when we remove an edge. Also observe that if (u,v) is an edge, then the graph obtained by contracting (u,v) to one vertex is also planar. Any graph H that can be obtained by removing and contracting edges from a graph G is called a *minor* of G. It is easy to show that every minor of a planar graph is also planar. Kuratowski's Theorem tells us that a graph is planar *if and only if* it does not have K_5 or $K_{3,3}$ (the complete bipartite graph between two sets of 3 vertices) as a minor. We will just use the fact that a planar graph does not have $K_{3,3}$ as a minor.

26.6 $\mathsf{cdv}(G) \leq 3$

We will now prove that if G is a 3-connected planar graph, then $\mathsf{cdv}(G) \leq 3$. Assume, by way of contradiction, that there is generalized Laplacian matrix M of G whose second eigenvalue λ_2 has multiplicity greater than or equal to 4. We will do this by showing that if G is three-connected and $\mathsf{cdv}(G) \geq 4$, then G contains a $K_{3,3}$ minor. Without loss of generality, we can assume $\lambda_2 = 0$ (by just adding a diagonal matrix).

Our proof will exploit a variant of Fiedler's Nodal Domain Theorem, which we proved back in the beginning of the semester. That theorem considered *any* eigenvector $\boldsymbol{v}$ of λ_2 (of a Laplacian), and proved that the set of vertices that are non-negative in $\boldsymbol{v}$ is connected. The variant we use is due to van der Holst [Van95], which instead applies to eigenvectors $\boldsymbol{v}$ of λ_2 of *minimal support*. These are the eigenvectors of $\boldsymbol{v}$ of λ_2 for which there is no other eigenvector $\boldsymbol{w}$ of λ_2 such that the zeros of $\boldsymbol{v}$ are a subset of the zeros of $\boldsymbol{w}$. That is, $\boldsymbol{v}$ has as many zero entries as possible. One can then prove that the set of vertices that are positive in $\boldsymbol{v}$ is connected. And, one can of course do the same for the vertices that are negative.

Now, let F be any face of G, and let a, b and c be three vertices in F. As λ_2 has multiplicty at least 4, it has some eigenvector that is zero at each of a, b and c. Let $\boldsymbol{v}$ be an eigenvector of λ_2 with minimal support that is zero at each of a, b, and c. Let d be any vertex for which $\boldsymbol{v}(d) > 0$. As the graph is three-connected, it contains three vertex-disjoint paths from d to a, b, and c (this follows from Menger's Theorem, which I have not covered). As $\boldsymbol{v}(d) > 0$ and $\boldsymbol{v}(a) = 0$, there is some vertex a' on the path from d to a for which $\boldsymbol{v}(a') = 0$ but a' has a neighbor a^+ for which $\boldsymbol{v}(a^+) > 0$. As $\lambda_2 = 0$, a' must also have a neighbor a^- for which $\boldsymbol{v}(a^-) < 0$. Construct similar vertices for b and c.

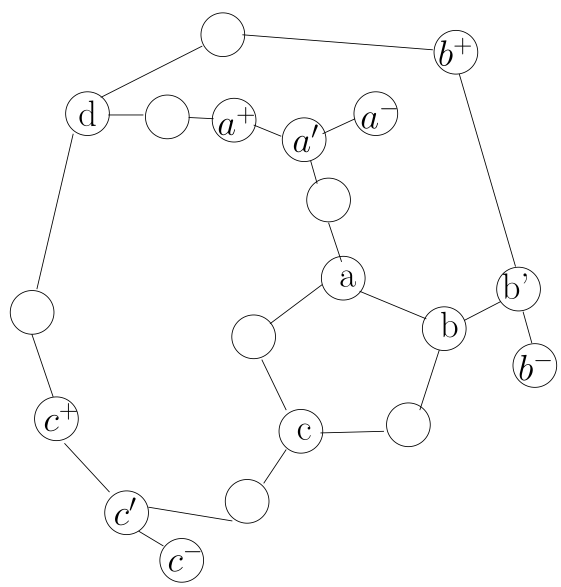

Figure 26.1: Vertices a, b, c, d, and the paths.

Now, contract every edge on the path from a to a', on the path from b to b' and on the path from c to c'. Also, contract all the vertices for which $\boldsymbol{v}$ is positive and contract all the vertices for which $\boldsymbol{v}$ is negative (which we can do because these sets are connected). Finally, contract every edge in the face F that does not involve one of a, b, or c. We obtain a graph with a triangular face abc such that each of a, b, and c have an edge to the positive supervertex and the negative supervertex. We would like to say that this graph cannot be planar.

To do this, we add one additional vertex f inside the face and connected to each of a, b, and c. This does not violate planarity because a, b, and c were contained in a face. In fact, we can add f before we do the contractions. By throwing away all other edges, we have constructed a $K_{3,3}$ minor, so the graph cannot be planar.

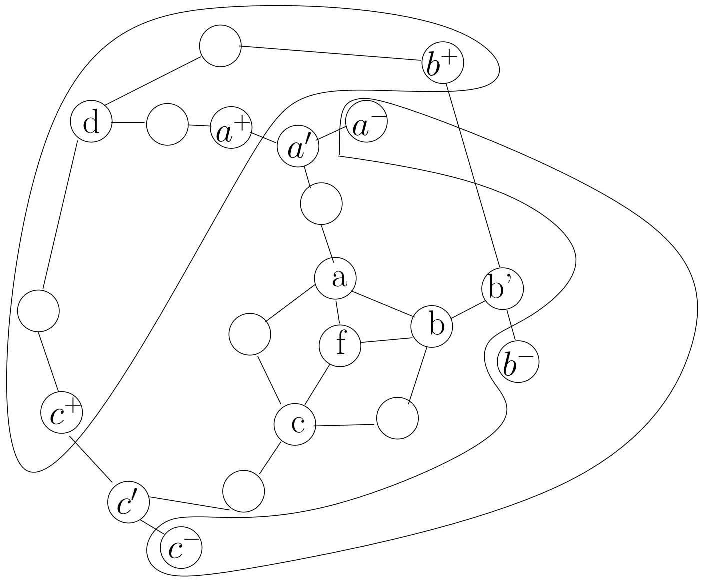

Figure 26.2: The set of positive and negative vertices that will be contracted. Vertex f has been inserted.

References

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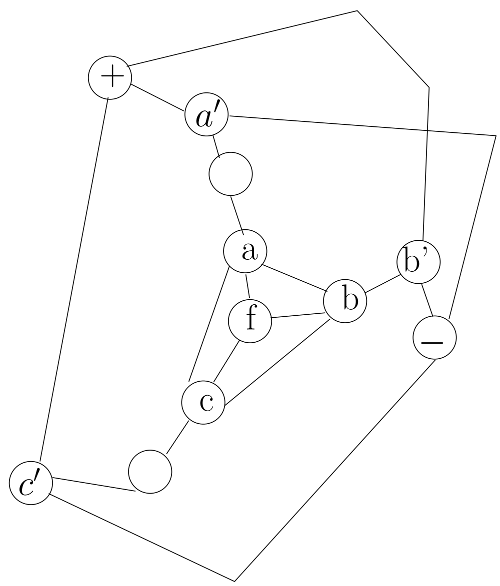

Figure 26.3: The edges in the cycle have been contracted, as have all the positive and negative vertices. After contracting the paths between a and a', between b and b' and between c and c', we obtain a $K_{3,3}$ minor.