**Spectral Graph Theory** Lecture 8

# Effective Resistance

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## 8.1 About these notes

These notes are not necessarily an accurate representation of what happened in class. The notes written before class say what I think I should say. The notes written after class way what I wish I said.

## 8.2 Overview

One of our main motivations for studying the Laplacian matrix is its role in the analysis of physical systems. I introduce this connection through the example of effective resistance. We will prove that effective resistance is a distance. It will greatly simplify the next lecture.

## 8.3 Resistor Networks

Given a graph, we can treat each edge as a resistor. If the graph is unweighted, we will assume that the resistor has resistance 1. If an edge $e$ has weight $w(e)$, we will give the corresponding resistor resistance $r(e) = 1/w(e)$. The reason is that when the weight of an edge is very small, the edge is barely there, so it should correspond to very high resistance. Having no edge corresponds to having a resistor of infinite resistance.

The first equation I recall is

$$V = IR,$$

which says that the potential drop across a resistor is equal to the current flowing over the resistor times the resistance. To apply this in a graph, we will define for each edge $(a, b)$ the current flowing *from $a$ to $b$* to be $\boldsymbol{i}(a, b)$. As this is a directed quantity, we define

$$\boldsymbol{i}(b, a) = -\boldsymbol{i}(a, b).$$

I now let $\boldsymbol{v} \in \mathbb{R}^V$ be the vector of potentials at vertices. Given these potentials (voltages), we can figure out how much current flows on each edge by the formula

$$\boldsymbol{i}(a, b) = \frac{1}{r_{a,b}} \left(\boldsymbol{v}(a) - \boldsymbol{v}(b)\right) = w_{a,b} \left(\boldsymbol{v}(a) - \boldsymbol{v}(b)\right).$$

That is, we adopt the convention that current flows from high voltage to low voltage. I would now like to write this equation in matrix form. The one complication is that each edge comes up twice in $\boldsymbol{i}$. So, to treat $\boldsymbol{i}$ as a vector I will have each edge show up exactly once as $(a,b)$ when $a < b$. I now define the *signed edge-vertex adjacency matrix* of the graph $\boldsymbol{U}$ to be the matrix with rows indexed by edges, columns indexed by vertices, such that

$$\boldsymbol{U}((a,b),c) = \begin{cases} 1 & \text{if } a = c \\ -1 & \text{if } b = c \\ 0 & \text{otherwise.} \end{cases}$$

Define $\boldsymbol{W}$ to be the diagonal matrix with rows and columns indexed by edges and the weights of edges on the diagonals. We then have

$$\boldsymbol{i} = \boldsymbol{W}\boldsymbol{U}\boldsymbol{v}.$$

Also recall that resistor networks cannot hold current. So, all the flow entering a vertex $a$ from edges in the graph must exit $a$ to an external source. Let $\boldsymbol{i}_{ext} \in \mathbb{R}^V$ denote the external currents, where $\boldsymbol{i}_{ext}(a)$ is the amount of current entering the graph through node $a$. We then have

$$\boldsymbol{i}_{ext}(a) = \sum_{b:(a,b)\in E} \boldsymbol{i}(a,b).$$

In matrix form, this becomes

$$\boldsymbol{i}_{ext} = \boldsymbol{U}^T \boldsymbol{i} = \boldsymbol{U}^T \boldsymbol{W}\boldsymbol{U}\boldsymbol{v}. \tag{8.1}$$

The matrix

$$\boldsymbol{L} \stackrel{\text{def}}{=} \boldsymbol{U}^T \boldsymbol{W}\boldsymbol{U}$$

is, of course, the Laplacian.

It is often helpful to think of the nodes $a$ for which $\boldsymbol{i}_{ext}(a) \neq 0$ as being *boundary* nodes. We will call the other nodes *internal*. Let's see what the equation

$$\boldsymbol{i}_{ext} = \boldsymbol{L}\boldsymbol{v}.$$

means for the internal nodes. If the graph is unweighted and $a$ is an internal node, then the $a$th row of this equation is

$$0 = \boldsymbol{L}(a,\cdot)\boldsymbol{v} = \sum_{(a,b)\in E} (\boldsymbol{v}(a) - \boldsymbol{v}(b)) = d(a)\boldsymbol{v}(a) - \sum_{(a,b)\in E} \boldsymbol{v}(b).$$

That is,

$$\boldsymbol{v}(a) = \frac{1}{d(a)} \sum_{(a,b)\in E} \boldsymbol{v}(b),$$

which means that the value of $\boldsymbol{v}$ at $a$ is the average of the values of $\boldsymbol{v}$ at the neighbors of $a$. We find a similar relation in a weighted graph, except that it tells us that the value of $\boldsymbol{v}$ at $a$ is a weighted average of the values of $\boldsymbol{v}$ at the neighbors of $a$.

We are often interested in applying (8.1) in the reverse: given a vector of external currents $\boldsymbol{i}_{ext}$ we solve for the induced voltages by

$$\boldsymbol{v} = \boldsymbol{L}^{-1} \boldsymbol{i}_{ext}.$$

This at first appears problematic, as the Laplacian matrix does not have an inverse. The way around this problem is to observe that we are only interested in solving these equations for vectors $\boldsymbol{i}_{ext}$ for which the system has a solution. In the case of a connected graph, this equation will have a solution if the sum of the values of $\boldsymbol{i}_{ext}$ is zero. That is, if the current going in to the circuit equals the current going out.

To obtain the solution to this equation, we multiply $\boldsymbol{i}_{ext}$ by the Moore-Penrose *pseudo-inverse* of $\boldsymbol{L}$.

**Definition 8.3.1.** *The pseudo-inverse of a symmetric matrix $\boldsymbol{L}$, written $\boldsymbol{L}^{+}$, is the matrix that has the same span as $\boldsymbol{L}$ and that satisfies*

$$\boldsymbol{L}\boldsymbol{L}^{+} = \mathbf{\Pi},$$

*where $\mathbf{\Pi}$ is the symmetric projection onto the span of $\boldsymbol{L}$.*

The symmetric case is rather special. As $\boldsymbol{L}\mathbf{\Pi} = \boldsymbol{L}$, the other following properties of the Moore-Penrose pseudo inverse follow from this one:

$$\begin{aligned} \boldsymbol{L}^{+}\boldsymbol{L} &= \mathbf{\Pi}, \\ \boldsymbol{L}\boldsymbol{L}^{+}\boldsymbol{L} &= \boldsymbol{L} \\ \boldsymbol{L}^{+}\boldsymbol{L}\boldsymbol{L}^{+} &= \boldsymbol{L}^{+}. \end{aligned}$$

It is easy to find a formula for the pseudo-inverse. First, let $\boldsymbol{\Psi}$ be the matrix whose $i$th column is $\boldsymbol{\psi}_i$ and let $\boldsymbol{\Lambda}$ be the diagonal matrix with $\lambda_i$ on the *ith* diagonal.

**Claim 8.3.2.**

$$\boldsymbol{L} = \boldsymbol{\Psi}\boldsymbol{\Lambda}\boldsymbol{\Psi}^{T} = \sum_i \lambda_i \boldsymbol{\psi}_i \boldsymbol{\psi}_i^{T}.$$

*Proof.* It suffices to check that both the left and right hand sides give the same expression when multiplied by $\boldsymbol{\psi}_i$ for each $i$. We have

$$\boldsymbol{\Psi}\boldsymbol{\Lambda}\boldsymbol{\Psi}^{T}\psi_i = \boldsymbol{\Psi}\boldsymbol{\Lambda}\boldsymbol{e}_i = \lambda_i \boldsymbol{\Psi}\boldsymbol{e}_i = \lambda_i \boldsymbol{\psi}_i.$$

□

That is one of the fundamental results of spectral theory. I'm surprised that I haven't used it yet in this class.

**Lemma 8.3.3.**

$$\boldsymbol{L}^{+} = \sum_{i>1} (1/\lambda_i) \boldsymbol{\psi}_i \boldsymbol{\psi}_i^{T}.$$

Before leaving this section, I wish to point out that

$$\boldsymbol{L}^2 = (\boldsymbol{\Psi} \boldsymbol{\Lambda} \boldsymbol{\Psi}^T)(\boldsymbol{\Psi} \boldsymbol{\Lambda} \boldsymbol{\Psi}^T) = \boldsymbol{\Psi} \boldsymbol{\Lambda}^2 \boldsymbol{\Psi}^T,$$

as $\boldsymbol{\Psi}^T \boldsymbol{\Psi} = I$. In general, we have

$$\boldsymbol{L}^k = \boldsymbol{\Psi} \boldsymbol{\Lambda}^k \boldsymbol{\Psi}^T = \sum_i \lambda_i^k \boldsymbol{\psi}_i \boldsymbol{\psi}_i^T.$$

Moreover, this holds for any symmetric matrix. Not just Laplacians.

## 8.4 Examples

We first consider a unit flow from one end to the other of a path graph with $n$ vertices and edges of resistance $r(1,2), \ldots, r(n-1,n)$. One way to achieve a flow of 1 is to set the potential of vertex $n$ to 0 and for every other vertex $i$ by

$$\boldsymbol{v}(i) = r(i, i+1) + \cdots + r(n-1, n).$$

Ohm's law then tells us that the current flow over the edge $(i, i+1)$ will be

$$\left(\boldsymbol{v}(i) - \boldsymbol{v}(i+1)\right) / r(i, i+1) = 1.$$

Note that the total potential difference between vertex 1 and vertex $n$ is the sum of the resistances of the edges in between.

## 8.5 Effective Resistance

The effective resistance between vertices $a$ and $b$ is the resistance between $a$ and $b$ given by the whole network. That is, if we treat the entire network as a resistor.

To figure out what this is, recall the equation

$$\boldsymbol{i}(a,b) = \frac{\boldsymbol{v}(a) - \boldsymbol{v}(b)}{r_{a,b}},$$

which holds for one resistor. We define effective resistance through this equation. That is, we consider an electrical flow that sends one unit of current into node $a$ and removes one unit of current from node $b$. We then measure the potential difference between $a$ and $b$ that is required to realize this current, define this to be the effective resistance between $a$ and $b$, and write it $\mathrm{R_{eff}}(a,b)$.

Algebraically, define $\boldsymbol{i}_{ext}$ to be the vector

$$\boldsymbol{i}_{ext}(c) = \begin{cases} 1 & \text{if } c = a \\ -1 & \text{if } c = b \\ 0 & \text{otherwise} \end{cases}.$$

This corresponds to a flow of 1 from $a$ to $b$. We then solve for the voltages that realize this flow:

$$\boldsymbol{L}\boldsymbol{v} = \boldsymbol{i}_{ext},$$

by

$$\boldsymbol{v} = \boldsymbol{L}^{+}\boldsymbol{i}_{ext}.$$

We thus have

$$\boldsymbol{v}(a) - \boldsymbol{v}(b) = \boldsymbol{i}_{ext}^{T}\boldsymbol{v} = \boldsymbol{i}_{ext}^{T}\boldsymbol{L}^{+}\boldsymbol{i}_{ext}.$$

Let's also observe that the solution is not unique. As $\mathbf{1}$ is in the nullspace of $\boldsymbol{L}$,

$$\boldsymbol{L}(\boldsymbol{v} + c\mathbf{1}) = \boldsymbol{L}\boldsymbol{v} + c\boldsymbol{L}\mathbf{1} = \boldsymbol{L}\boldsymbol{v}$$

for every constant $c$.

Recalling that $\boldsymbol{L}\boldsymbol{v} = \boldsymbol{i}_{ext}$, we obtain

$$\boldsymbol{i}_{ext}^{T}\boldsymbol{L}^{+}\boldsymbol{i}_{ext} = \boldsymbol{v}^{T}\boldsymbol{L}\boldsymbol{L}^{+}\boldsymbol{L}\boldsymbol{v} = \boldsymbol{v}^{T}\boldsymbol{L}\boldsymbol{v}.$$

## 8.6 Effective Resistance as a Distance

A distance is any function on pairs of verices such that

1. $\delta(a, a) = 0$ for every vertex $a$,
2. $\delta(a, b) \geq 0$ for all vertices $a, b$,
3. $\delta(a, b) = \delta(b, a)$, and
4. $\delta(a, c) \leq \delta(a, b) + \delta(b, c)$.

We claim that the effective resistance is a distance. The only non-trivial part to prove is the triangle inequality, (4).

Our proof will exploit a trivial but important observation about the voltages induced by a unit flow between $a$ and $b$.

**Claim 8.6.1.** *Let $\boldsymbol{i}_{a,b}$ denote the vector that is $1$ at $a$, $-1$ at $b$, and zero elsewhere. Let $\boldsymbol{v} = \boldsymbol{L}^{+}\boldsymbol{i}_{a,b}$. Then, for every vertex $c$,*

$$\boldsymbol{v}(a) \geq \boldsymbol{v}(c) \geq \boldsymbol{v}(b).$$

*Proof.* We know that every vertex $c$ other than $a$ and $b$ has a value under $\boldsymbol{v}$ that is equal to a weighted average of the values of $\boldsymbol{v}$ at its neighbors. We can use this to prove the assertion by contradiction. For example, assume that there is a vertex $c$ for which $\boldsymbol{v}(c) > \boldsymbol{v}(b)$. Then, let $c$ be the vertex at which $\boldsymbol{v}(c)$ is largest. If there is only one vertex $c$ with this value $\boldsymbol{v}(c)$, and if this value exceeds $\boldsymbol{v}(b)$, then it is clear that the value of $\boldsymbol{v}$ at $c$ is more than the average of its neighbors. If there are many vertices that achieve the value $\boldsymbol{v}(c)$, then we can repeat this argument with their sum. As long at the graph is connected they will have an edge to a vertex of lower value, and so cannot all be weighted averages of the values at their neighbors. $\square$

**Lemma 8.6.2.** *Let $a$, $b$ and $c$ be vertices in a graph. Then*

$$\mathrm{R_{eff}}(a,b) + \mathrm{R_{eff}}(b,c) \geq \mathrm{R_{eff}}(a,c).$$

*Proof.* Let $\boldsymbol{i}_{a,b}$ be the external current corresponding to sending one unit of current from $a$ to $b$, and let $\boldsymbol{i}_{b,c}$ be the external current corresponding to sending one unit of current from $b$ to $c$. Note that

$$\boldsymbol{i}_{a,c} = \boldsymbol{i}_{a,b} + \boldsymbol{i}_{b,c}.$$

Now, define the corresponding voltages by

$$\boldsymbol{v}_{a,b} = \boldsymbol{L}^{+}\boldsymbol{i}_{a,b} \quad \boldsymbol{v}_{b,c} = \boldsymbol{L}^{+}\boldsymbol{i}_{b,c}. \quad \boldsymbol{v}_{a,c} = \boldsymbol{L}^{+}\boldsymbol{i}_{a,c}.$$

By linearity, we have

$$\boldsymbol{v}_{a,c} = \boldsymbol{v}_{a,b} + \boldsymbol{v}_{b,c},$$

and so

$$\mathrm{R_{eff}}(a,c) = \boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{a,c} = \boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{a,b} + \boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{b,c}.$$

By Claim 8.6.1, we have

$$\boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{a,b} = \boldsymbol{v}_{a,b}(a) - \boldsymbol{v}_{a,b}(c) \leq \boldsymbol{v}_{a,b}(a) - \boldsymbol{v}_{a,b}(b) = \mathrm{R_{eff}}(a,b)$$

and similarly

$$\boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{b,c} \leq \mathrm{R_{eff}}(b,c).$$

The lemma follows. □

## 8.7 Fixing Potentials

The analysis of the last section might seem a little artifical because one doesn't typically fix the flow going into a circuit. Rather, one typically fixes the potentials of the boundary vertices by attaching them to the terminals of a battery. Intuitively, this should give another way of defining the effective resistance between vertices $a$ and $b$: if we compute the electrical flow obtained when we fix $\boldsymbol{v}(a) = 1$, $\boldsymbol{v}(b) = 0$, and force the external current at every other vertex to be zero, then the effective resistance should be the reciprocal of the current that flows from $a$ to $b$. We will now see that we can realize this situation.

First, let $\boldsymbol{v}_{a,b}$ be the voltages of any electrical flow of one unit from $a$ to $b$. Now, set

$$\boldsymbol{w} = \frac{\boldsymbol{v}_{a,b} - \boldsymbol{v}_{a,b}(b)\mathbf{1}}{\boldsymbol{v}_{a,b}(a) - \boldsymbol{v}_{a,b}(b)}.$$

We can see that $\boldsymbol{w}$ provides the voltages of the desired flow. Shifing each voltage by subtracting $\boldsymbol{v}_{a,b}(b)$ does not change the flow on any edge. So, the only boundary vertices are $a$ and $b$, and for every vertex $c$ other than $a$ and $b$, $\boldsymbol{w}(c)$ is the weighted average of $\boldsymbol{w}$ over the neighbors of $c$. However, we have divided by $\boldsymbol{v}_{a,b}(a) - \boldsymbol{v}_{a,b}(b) = \mathrm{R_{eff}}(a,b)$, so the current that flows is $1/\mathrm{R_{eff}}(a,b)$. Finally, it is easy to check that $\boldsymbol{w}(a) = 1$ and $\boldsymbol{w}(b) = 0$.

Let's take a look at how we might compute such a vector $\boldsymbol{w}$. We can list $n$ constraints that the vector $\boldsymbol{w}$ should satisfy:

$$\begin{aligned}\boldsymbol{w}(a) &= 1,\\ \boldsymbol{w}(b) &= 0, \text{and}\\ \boldsymbol{w}(c) &= \frac{1}{d(c)} \sum_{(c,z)\in E} \boldsymbol{w}(z), \text{for } c \notin \{a,b\}.\end{aligned}$$

We already know that these equations have a solution, and it should be intuitively clear that it is unique if the graph is connected. But, we'll make sure by examining the matrix of these equations. For simplicity, assume that $a = 1$ and $b = 2$. Then, we can describe these equations in the form $\boldsymbol{M}\boldsymbol{w} = \boldsymbol{b}$, where

$$\begin{aligned}\boldsymbol{M}(1,:) &= \boldsymbol{e}_1^T \text{ and } \boldsymbol{b}(1) = 1,\\ \boldsymbol{M}(2,:) &= \boldsymbol{e}_2^T \text{ and } \boldsymbol{b}(2) = 0, \text{ and}\\ \boldsymbol{M}(c,:) &= \boldsymbol{L}(c,:), \text{for } c \notin \{1,2\}.\end{aligned}$$

This matrix $\boldsymbol{M}$ looks like $\boldsymbol{L}$, except that the rows corresponding to vertices $a$ and $b$ have been replaced by elementary unit vectors.

When faced with a matrix like this, one's natural reaction is to simplify the equations by using the first two rows to eliminate all of the non-zero entries in the first two columns. This yields a matrix like

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \boldsymbol{0}_{2,n-2} \\ \boldsymbol{0}_{n-2,2} & \boldsymbol{L}(3:n, 3:n) \end{pmatrix}.$$

We have reduced the problem to that of solving a system of equations in a submatrix of the Laplacian.

Submatrices of Laplacians are a lot like Laplacians, except that they are positive definite. To see this, note that all of the off-diagonals of the submatrix of $\boldsymbol{L}$ agree with all the off-diagonals of the Laplacian of the induced subgraph on the internal vertices. But, some of the diagonals are larger: the diagonals of nodes in the submatrix account for both edges in the induced subgraph and edges to the vertices $a$ and $b$.

**Lemma 8.7.1.** *Let $\boldsymbol{L}$ be the Laplacian matrix of a connected graph and let $\boldsymbol{X}$ be a non-negative, non-zero, diagonal matrix. Then, $\boldsymbol{L} + \boldsymbol{X}$ is positive definite.*

*Proof.* We will prove that $\boldsymbol{x}^T(\boldsymbol{L}+\boldsymbol{X})\boldsymbol{x} > 0$ for every nonzero vector $\boldsymbol{x}$. As $\boldsymbol{X}$ is positive semidefinite,

$$\boldsymbol{x}^T(\boldsymbol{L}+\boldsymbol{X})\boldsymbol{x} \geq \boldsymbol{x}^T\boldsymbol{L}\boldsymbol{x}.$$

This quadratic form is positive unless $\boldsymbol{x}$ is a constant vector. In the case of constant vectors, we obtain

$$\boldsymbol{1}^T(\boldsymbol{L}+\boldsymbol{X})\boldsymbol{1} = \boldsymbol{1}^T\boldsymbol{X}\boldsymbol{1} = \sum_i \boldsymbol{X}(i,i) > 0.$$

$\square$

We conclude that the matrix $\boldsymbol{M}$ non-singular.

## 8.8 A Positive Inverse

We now point out a remarkable property of submatrices of Laplacian matrices: every entry of their inverses is positive.

**Theorem 8.8.1.** *Let $\boldsymbol{L}$ be the Laplacian matrix of a connected graph and let $\boldsymbol{X}$ be a non-negative, non-zero, diagonal matrix. Then every entry of $(\boldsymbol{L}+\boldsymbol{X})^{-1}$ is positive.*

Our proof of this theorem will rely on a few simple statements. The first, a variation of Lemma 8.7.1, will appear on Problem Set 2.

**Lemma 8.8.2.** *Let $\boldsymbol{A}$ be the adjacency matrix of a connected graph and let $\boldsymbol{D}$ be its matrix of degrees (so $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$). If $\boldsymbol{X}$ is a non-negative, non-zero, diagonal matrix, then $\boldsymbol{X} + \boldsymbol{D} + \boldsymbol{A}$ is positive definite.*

*Proof of Theorem 8.8.1.* Let $\boldsymbol{D}$ be the diagonal of $\boldsymbol{L}+\boldsymbol{X}$. Consider the matrix

$$\boldsymbol{M} = \boldsymbol{D}^{-1/2}(\boldsymbol{L}+\boldsymbol{X})\boldsymbol{D}^{-1/2}.$$

We can write this matrix in the form $\boldsymbol{I} - \boldsymbol{A}$, where $\boldsymbol{A}$ is a non-negative matrix. As $\boldsymbol{L}+\boldsymbol{X}$ is postive definite, $\boldsymbol{M}$ is as well. So, the the largest eigenvalue of $\boldsymbol{A}$ is strictly less than 1. With a little thought, you can use Lemma 8.8.2 to see that that $\boldsymbol{I} + \boldsymbol{A}$ is positive definte too, and to thereby conclude that every eigenvalue of $\boldsymbol{A}$ is greater than $-1$. We can now apply Lemma 8.8.3 to show that

$$\boldsymbol{M}^{-1} = (\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{j \geq 0} \boldsymbol{A}^j.$$

This is clearly a non-negative matrix. One can use the fact that the underlying graph is connected to prove that it is in fact a positive matrix. As

$$(\boldsymbol{L}+\boldsymbol{X})^{-1} = \boldsymbol{D}^{1/2}\boldsymbol{M}^{-1}\boldsymbol{D}^{1/2},$$

every entry of $(\boldsymbol{L}+\boldsymbol{X})^{-1}$ is positive as well. □

**Lemma 8.8.3.** *Let $\boldsymbol{A}$ be a symmetric matrix with eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. If $\mu_n < 1$ and $\mu_1 > -1$, then*

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{j \geq 0} \boldsymbol{A}^j.$$

*Proof.* The key to this proof is really to show that the sum converges.

Let the eigenvectors of $\boldsymbol{A}$ be $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n$. We have already established that

$$\boldsymbol{A} = \sum_{i=1}^{n} \mu_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T.$$

So,

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \sum_{i=1}^{n} (1-\mu_i)^{-1} \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T.$$

As $-1 < \mu_i < 1$,

$$(1 - \mu_i)^{-1} = \sum_{j \geq 0} \mu_i^j,$$

and

$$\sum_{i=1}^{n} (1 - \mu_i)^{-1} \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T = \sum_{i=1}^{n} \sum_{j \geq 0} \mu_i^j \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T = \sum_{j \geq 0} \sum_{i=1}^{n} \mu_i^j \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T.$$

As

$$\boldsymbol{A}^j = \sum_{i=1}^{n} \mu_i^j \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T,$$

this sum is really

$$\sum_{j \geq 0} \boldsymbol{A}^j.$$

□