Spectral Graph Theory Lecture 16

The Simplest Construction of Expanders

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16.1 About these notes

These notes are not necessarily an accurate representation of what happened in class. The notes written before class say what I think I should say. The notes written after class way what I wish I said.

During class I lost a factor of 2. I found it while I was editing these notes.

16.2 Overview

I am going to present the simplest construction of expanders that I have been able to find. By "simplest", I mean optimizing the tradeoff of simplicity of construction with simplicty of analysis. It is inspired by the Zig-Zag product and replacement product constructions presented by Reingold, Vadhan and Wigderson [RVW02].

For those who want the quick description, here it is. Begin with an expander. Take its line graph. Obseve that the line graph is a union of cliques. So, replace each clique by a small expander. We need to improve the expansion slightly, so square the graph. Square one more time. Repeat.

The analysis will be simple because all of the important parts are equalities, which I find easier to understand than inequalities.

16.3 Line Graphs

Our construction will leverage small expanders to make bigger expanders. To begin, we need a way to make a graph bigger and still say something about its spectrum.

We use the *line graph* of a graph. Let $G = (V, E)$ be a graph. The line graph of G is the graph whose vertices are the edges of G in which two are connected if they share an endpoint in G. That is, $\big((u, v), (w, z)\big)$ is an edge of the line graph if one of $\{u, v\}$ is the same as one of $\{w, z\}$. The line graph is often written $L(G)$, but we won't do that in this class so that we can avoid confusion with the Laplacian.

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Let G be a d-regular graph with n vertices, and let H be its line graph. As G has $dn/2$ edges, H

has $dn/2$ vertices. Each vertex of H, say (u, v), has degree $2(d-1)$: $d-1$ neighbors for the other edges attached to u and $d-1$ for v. In fact, if we just consider one vertex u in V, then all vertices in H of form (u, v) of G will be connected. That is, H contains a d-clique for every vertex in V. We see that each vertex of H is contained in exactly two of these cliques.

Here is the great fact about the spectrum of the line graph.

Lemma 16.3.1. *Let G be a d-regular graph with n vertices, and let H be its line graph. Then the spectrum of the Laplacian of H is the same as the spectum of the Laplacian of G, except that it has $dn/2 - n$ extra eigenvalues of $2d$.*

Before we prove this lemma, we need to recall the factorization of a Laplacian as the product of the signed vertex-edge adjacency matrix times its transpose that we derived in Lecture 8. We reserved the letter $\boldsymbol{U}$ for this matrix, and defined it by

$$\boldsymbol{U}(c,(a,b)) = \begin{cases} 1 & \text{if } a = c \\ -1 & \text{if } b = c \\ 0 & \text{otherwise.} \end{cases}$$

For an unweighted graph, we have

$$\boldsymbol{L}_G = \boldsymbol{U}\boldsymbol{U}^T.$$

Recall that each edge indexes one column, and that we made an arbitrary choice when we ordered the edge (a, b) rather than (b, a). But, this arbitrary choice factors out when we multiply by $\boldsymbol{U}^T$.

16.4 The Spectrum of the Line Graph

Define the matrix $|\boldsymbol{U}|$ to be the matrix obtained by replacing every entry of $\boldsymbol{U}$ by its absolute value. Now, consider $|\boldsymbol{U}|\,|\boldsymbol{U}|^T$. It looks just like the Laplacian, except that all of its off-diagonal entries are 1 instead of -1. So,

$$|\boldsymbol{U}|\,|\boldsymbol{U}|^T = \boldsymbol{D}_G + \boldsymbol{A}_G = d\boldsymbol{I} + \boldsymbol{A}_G,$$

as G is d-regular. We will also consider the matrix $|\boldsymbol{U}|^T\,|\boldsymbol{U}|$. This is a matrix with $nd/2$ rows and $nd/2$ columns, indexed by edges of G. The entry at the intersection of row (u, v) and column (w, z) is

$$(\boldsymbol{e}_u + \boldsymbol{e}_v)^T(\boldsymbol{e}_w + \boldsymbol{e}_z).$$

So, it is 2 if these are the same edge, 1 if they share a vertex, and 0 otherwise. That is

$$|\boldsymbol{U}|^T\,|\boldsymbol{U}| = 2I_{nd/2} + A_H.$$

Moreover, $|\boldsymbol{U}|\,|\boldsymbol{U}|^T$ and $|\boldsymbol{U}|^T\,|\boldsymbol{U}|$ have the same eigenvalues, except that the later matrix has $nd/2 - n$ extra eigenvalues of 0.

Proof of Lemma 16.3.1. First, let λ_i be an eigenvalue of $\boldsymbol{L}_G$. We see that

$$\begin{aligned}
&\lambda_i \text{ is an eigenvalue of } \boldsymbol{D}_G - \boldsymbol{A}_G \implies \\
&d - \lambda_i \text{ is an eigenvalue of } \boldsymbol{A}_G \implies \\
&2d - \lambda_i \text{ is an eigenvalue of } \boldsymbol{D}_G + \boldsymbol{A}_G \implies \\
&2d - \lambda_i \text{ is an eigenvalue of } 2I_{nd/2} + \boldsymbol{A}_H \implies \\
&2(d-1) - \lambda_i \text{ is an eigenvalue of } \boldsymbol{A}_H \implies \\
&\lambda_i \text{ is an eigenvalue of } \boldsymbol{D}_H - \boldsymbol{A}_H.
\end{aligned}$$

Of course, this last matrix is the Laplacian matrix of H. We can similarly show that the extra $dn/2 - n$ zero eigenvalues of $2I_{nd/2} + \boldsymbol{A}_H$ become $2d$ in $\boldsymbol{L}_H$. □

While the line graph operation preserves λ_2, it causes the degree of the graph to grow. So, we are going to need to do more than just take line graphs to construct expanders.

To measure the qualities of the graphs that appear in our construction, we define a quantity that we will call the *spectral ratio* of a graph that we define by

$$r(G) \stackrel{\text{def}}{=} \min\left(\frac{\lambda_2(G)}{d}, \frac{2d - \lambda_n}{d}\right).$$

The graphs with larger spectral ratios are better expanders. An ϵ-expander has spectral ratio at least $1 - \epsilon$.

Proposition 16.4.1. *Let G be a d-regular graph for $d \geq 6$ and let H be its line graph. Then,*

$$r(H) = \frac{\lambda_2(G)}{2(d-1)} \geq r(G)/2.$$

Proof. As G is d-regular, $\lambda_2(G) \leq d$ and $\lambda_{max}(G) \leq 2d$. So, $\lambda_{max}(H) = 2d$ and $\lambda_2(H) = \lambda_2(G) \leq d$. So, the term in the definition of the spectral ratio corresponding to the largest eigenvalue of H satisfies

$$\frac{2(2d-2) - \lambda_{max}(H)}{2d-2} = \frac{2(2d-2) - 2d}{2d-2} = 1 - \frac{2}{d} \geq 2/3,$$

as $d \geq 6$. On the other hand,

$$\frac{\lambda_2(H)}{2d-2} \leq \frac{d}{2d-2} \leq 3/5.$$

So,

$$\min\left(\frac{\lambda_2(H)}{2d-2}, \frac{2(2d-2) - \lambda_{max}(H)}{2d-2}\right) = \frac{\lambda_2(H)}{2d-2}.$$

□

We see that the spectral ratio of the line graph of G is approximately half that of G. But, the line graph has more vertices. Our construction will produce an infinite family of d-regular graphs with spectral ratio bounded below by some absolute constant $\beta > 0$. We will do this for small β, as the analysis for large β is trickier.

16.5 Approximations of Line Graphs

Our next step will be to construct approximations of line graphs. We already know how to approximate complete graphs: we use expanders. As line graphs are sums of complete graphs, we will approximate them by sums of expanders. That is, we replace each clique in the line graph by an expander on d vertices.

Let G be a d-regular graph and let Z be a graph on d vertices (we will use a low-degree expander). We define the graph

$$G \textcircled{L} Z$$

to be the graph obtained by forming the edge graph of G, H, and then replacing every d-clique in H by a copy of Z. Actually, this does not uniquely define $G \textcircled{L} Z$, as there are many ways to replace a d-clique by a copy of Z. But, any choice will work. Note that every vertex of $G \textcircled{L} Z$ has degree $2z$.

Lemma 16.5.1. *Let G be a d-regular graph, let H be the line graph of G, and let Z be a z-regular ϵ-expander. Then,*

$$(1-\epsilon)\frac{z}{d} H \preccurlyeq G \textcircled{L} Z \preccurlyeq (1+\epsilon)\frac{z}{d} H$$

Proof. As H is a sum of d-cliques, let $H_1, \dots, H_n$ be those d-cliques. So,

$$\boldsymbol{L}_H = \sum_{i=1}^{n} \boldsymbol{L}_{H_i}.$$

Let Z_i be the graph obtained by replacing H_i with a copy of Z, on the same set of vertices. To prove the lower bound, we compute

$$\boldsymbol{L}_{G \textcircled{L} Z} = \sum_{i=1}^{n} \boldsymbol{L}_{Z_i} \succcurlyeq (1-\epsilon)\frac{k}{d} \sum_{i=1}^{n} \boldsymbol{L}_{H_i} = (1-\epsilon)\frac{k}{d} \boldsymbol{L}_H.$$

The upper bound is proved similarly. □

Corollary 16.5.2. *Under the conditions of Lemma 16.5.1,*

$$r(G \textcircled{L} Z) \geq \frac{1-\epsilon}{2} r(G).$$

Proof. The proof is similar to the proof of Proposition 16.4.1. We have

$$\lambda_2(G \textcircled{L} Z) \geq (1-\epsilon)\frac{k\lambda_2(G)}{d},$$

and

$$\lambda_{max}(G \textcircled{L} Z) \leq (1+\epsilon)2k.$$

So,

$$\min\left(\lambda_2(G \textcircled{L} Z), 2(2k) - \lambda_{max}(G \textcircled{L} Z)\right) \geq \min\left((1-\epsilon)\frac{k\lambda_2(G)}{d}, (1-\epsilon)2k\right) = (1-\epsilon)\frac{k\lambda_2(G)}{d},$$

as $\lambda_2(G) \leq d$. So,

$$r(G \textcircled{L} Z) \geq \frac{1}{2k}(1-\epsilon)kr(G) = \frac{1-\epsilon}{2} r(G).$$

□

So, the spectral ratio of $G \textcircled{L} Z$ is a little less than half that of G. But, the degree of $G \textcircled{L} Z$ is $2z$, which we will arrange to be much less than the degree of G, d.

16.6 Squaring the graph

We can improve the spectral ratio of a graph by squaring it, at the cost of increasing its degree. I recall the theorem we proved about squares of graphs in the previous lecture.

Lemma 16.6.1. *Let G be a d-regular graph with Laplacian eigenvalues is $\lambda_1, \ldots, \lambda_n$. Then, G^2 is a $d(d-1)$-regular graph with Laplacian eigenvalues*

$$2d\lambda_i - \lambda_i^2.$$

In particular, the largest Laplacian eigenvalue of G^2 is at most d^2.

We observed that if δ is small and the spectral ratio of G is δ, then the spectral ratio of G^2 is a little less than 2δ. In particular, if the spectral ratio of G is at least δ, then the spectral ratio of G^2 is at least $2\delta - \delta^2$.

16.7 The whole construction

To begin, we need a "small" z-regular expander graph Z on

$$d \stackrel{\text{def}}{=} (2z(2z-1))^2 - 2z(2z-1)$$

vertices. It should be an ϵ-expander for some small ϵ. I believe that $\epsilon = 1/6$ would suffice.

The other graph we will need to begin our construction will be a small d-regular expander graph G_0. We could use the constructions of expanders from error-correcting codes that we saw last lecture to define Z and G_0. And let β be the spectral ratio of G_0. We will assume that β is small, but greater than 0. I believe that $\beta = 1/5$ will work. Of course, it does not hurt to start with a graph of larger spectral ratio.

We then construct $G_0 \textcircled{L} Z$. The degree of this graph is $2z$, and its spectral ratio is a little less than $\beta/2$. So, we square the resulting graph, to obtain

$$(G_0 \textcircled{L} Z)^2.$$

It has degree approximately $4z^2$, and spectral ratio slightly less than β. But, for induction, we need it to be more than β. So, we square one more time, to get a spectral ratio a little less than 2β. We now set

$$G_1 = \left((G_0 \textcircled{L} Z)^2 \right)^2 .$$

As G_1 is a square, its largest Laplacian eigenvalue is extremely close to its degree. The graph G_1 is at least as good an approximation of a complete graph as G_0, and it has degree approximately $16z^4$. In general, we set

$$G_{i+1} = \left((G_i \textcircled{L} Z)^2 \right)^2 .$$

To make the inductive construction work, we need for Z to be a graph of degree z whose number of vertices equals the degree of G. This is approximately $16z^4$, and is exactly

$$(2z(2z-1))^2 - 2z(2z-1).$$

I'll now carry out the computation of spectral ratios with more care. Let's assume that G_0 has a spectral ratio of $\beta \geq 4/5$, and assume, by way of induction, that $\rho(G_i) \geq 4/5$. Also assume that Z is a $1/6$-expander. We then find

$$r(G_i \textcircled{L} Z) \geq (1-\epsilon)(4/5)/2 = 1/3.$$

So, $G_i \textcircled{L} Z$ is a $2/3$-expander. Our analysis of graph squares then tells us that G_{i+1} is a $(2/3)^4$-expander. So,

$$r(G_{i+1}) \geq 1 - (2/3)^4 = 65/81 > 4/5.$$

By induction, we conclude that every G_i has spectral ratio at least $4/5$.

To improve their spectral ratios of the graphs we produce, we can just square them a few times.

16.8 Better Constructions

There is a better construction technique, called the Zig-Zag product [RVW02]. The Zig-Zag construction is a little trickier to understand, but it achieves better expansion. I chose to present the line-graph based construction because its analysis is very closely related to an analysis of the Zig-Zag product.

References

[RVW02] Omer Reingold, Salil Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. *Annals of Mathematics*, 155(1):157–187, 2002.