

Spring and Resistor Networks

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Disclaimer

These notes are not necessarily an accurate representation of what happened in class. The notes written before class say what I think I should say. I sometimes edit the notes after class to make them say what I wish I had said.

There may be small mistakes, so I recommend that you check any mathematically precise statement before using it in your own work.

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7.1 Overview

This lecture is the first of two introducing systems of linear equations in Laplacian matrices and their submatrices. In later lectures we will see how to solve such systems of equations.

7.2 Spring Networks

We begin by imagining that every edge of a graph $G = (V, E)$ is an ideal spring or rubber band. They are joined together at the vertices. Given such a structure, we will pick a subset of the vertices $F \subseteq V$ and fix the location of every vertex in F . For example, you could nail each vertex in F onto a point in the real line, or onto a board in \mathbb{R}^2 . We will then study where the other vertices wind up.

We can use Hooke's law to figure this out. To begin, assume that each rubber band is an ideal spring with spring constant 1. If your graph is weighted, then the spring constant of each edge should be its weight. If a rubber band connects vertices u and v , then Hooke's law tells us that the force it exerts at node u is in the direction of v and is proportional to the distance between u and v . Let $\mathbf{x}(u)$ be the position of each vertex u . You should begin by thinking of $\mathbf{x}(u)$ being in \mathbb{R} , but you will see that it is just as easy to make it a vector in \mathbb{R}^2 or \mathbb{R}^k for any k .

The force the rubber band between u and v exerts on u is

$$\mathbf{x}(v) - \mathbf{x}(u).$$

In a stable configuration, all of the vertices that have not been nailed down must experience a zero net force. That is

$$\begin{aligned} \sum_{v:(u,v) \in E} \mathbf{x}(v) - \mathbf{x}(u) = 0 &\iff \sum_{v:(u,v) \in E} \mathbf{x}(v) = d(u)\mathbf{x}(u) \\ &\iff \frac{1}{d(u)} \sum_{v:(u,v) \in E} \mathbf{x}(v) = \mathbf{x}(u). \end{aligned}$$

In a stable configuration, every vertex that is not fixed must be the average of its neighbors.

In the weighted case, we would have for each $u \in V - F$

$$\frac{1}{d(u)} \sum_{v:(u,v) \in E} w_{u,v} \mathbf{x}(v) = \mathbf{x}(u), \quad (7.1)$$

where here

$$d(u) = \sum_{v:(u,v) \in E} w_{u,v}$$

is the weighted degree of node u .

A function \mathbf{x} that satisfies these equations for each vertex $u \in V - F$ is said to be *harmonic* on $V - F$.

We will next show that the equations (7.1) have a solution, and that it is unique¹ if the underlying graph is connected. But first, let's do a simple example.

Consider the path graph P_n with the endpoints fixed: $F = \{1, n\}$. Let us fix them to the values $\mathbf{f}(1) = 1$ and $\mathbf{f}(n) = n$. The only solution to the equations (7.1) is the obvious one: vertex i is mapped to i : $\mathbf{x}(i) = i$ for all i .

7.3 Laplacian linear equations

If we rewrite equation (7.1) as

$$d(u)\mathbf{x}(u) - \sum_{v:(u,v) \in E} w_{u,v} \mathbf{x}(v) = 0, \quad (7.2)$$

we see that it corresponds to the row of the Laplacian matrix corresponding to vertex u . So, we may find a solution to the equations (7.1) by solving a system of equations in the submatrix of the Laplacian indexed by vertices in $V - F$.

To be more concrete, I will set up those equations. For each vertex $u \in F$, let its position be fixed to $\mathbf{f}(u)$. Then, we can re-write equation (7.2) as

$$d(u)\mathbf{x}(u) - \sum_{v \notin F: (u,v) \in E} w_{u,v} \mathbf{x}(v) = \sum_{v \in F: (u,v) \in E} w_{u,v} \mathbf{f}(v),$$

¹It can only fail to be unique if there is a connected component that contains no vertices of F .

for each $u \in V - F$. So, all of the fixed terms wind up in the right-hand vector.

Let $S = V - F$. We now see that this is an equation of the form

$$\mathbf{L}(S, S)\mathbf{x}(S) = \mathbf{b},$$

where by $\mathbf{L}(S, S)$ I mean the submatrix of \mathbf{L} indexed by rows and columns of S , and by $\mathbf{x}(S)$ I mean the sub-vector of \mathbf{x} indexed by S . We can then write the condition that entries of F are fixed to \mathbf{f} by

$$\mathbf{x}(F) = \mathbf{f}(F).$$

We have reduced the problem to that of solving a system of equations in a submatrix of the Laplacian.

Submatrices of Laplacians are a lot like Laplacians, except that they are positive definite. To see this, note that all of the off-diagonals of the submatrix of \mathbf{L} agree with all the off-diagonals of the Laplacian of the induced subgraph on the internal vertices. But, some of the diagonals are larger: the diagonals of nodes in the submatrix account for both edges in the induced subgraph and edges to the vertices in F .

Lemma 7.3.1. *Let \mathbf{L} be the Laplacian matrix of a connected graph and let \mathbf{X} be a nonnegative, diagonal matrix with at least one nonzero entry. Then, $\mathbf{L} + \mathbf{X}$ is positive definite.*

Proof. We will prove that $\mathbf{x}^T(\mathbf{L} + \mathbf{X})\mathbf{x} > 0$ for every nonzero vector \mathbf{x} . As both \mathbf{L} and \mathbf{X} are positive semidefinite, we have

$$\mathbf{x}^T(\mathbf{L} + \mathbf{X})\mathbf{x} \geq \min(\mathbf{x}^T\mathbf{L}\mathbf{x}, \mathbf{x}^T\mathbf{X}\mathbf{x}).$$

As the graph is connected, $\mathbf{x}^T\mathbf{L}\mathbf{x}$ is positive unless \mathbf{x} is a constant vector. If $\mathbf{x} = c\mathbf{1}$ for some $c \neq 0$, then we obtain

$$c^2\mathbf{1}^T(\mathbf{L} + \mathbf{X})\mathbf{1} = c^2\mathbf{1}^T\mathbf{X}\mathbf{1} = c^2 \sum_i \mathbf{X}(i, i) > 0.$$

□

This tells us that the matrix $\mathbf{L}(S, S)$ is invertible. So, the equations have a solution, and it must be unique.

7.4 Energy

Physics also tells us that the vertices will settle into the position that minimizes the potential energy. The potential energy of an ideal linear spring with constant w when stretched to length l is

$$\frac{1}{2}wl^2.$$

So, the potential energy in a configuration \mathbf{x} is given by

$$\mathcal{E}(\mathbf{x}) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{(u,v) \in E} w_{u,v} (\mathbf{x}(u) - \mathbf{x}(v))^2. \quad (7.3)$$

For any \mathbf{x} that minimizes the energy, the partial derivative of the energy with respect to each variable must be zero. In this case, the variables are $\mathbf{x}(u)$ for $u \in V - W$. The partial derivative with respect to $\mathbf{x}(u)$ is

$$\frac{1}{2} \sum_{v:(u,v) \in E} w_{u,v} 2(\mathbf{x}(u) - \mathbf{x}(v)) = \sum_{v:(u,v) \in E} w_{u,v} (\mathbf{x}(u) - \mathbf{x}(v)).$$

Setting this to zero gives the equations we previously derived: (7.1).

7.5 Resistor Networks

We now study a related physical model of a graph in which we treat every edge as a resistor. If the graph is unweighted, we will assume that each resistor has resistance 1. If an edge e has weight w_e , we will give the corresponding resistor resistance $r_e = 1/w_e$. The reason is that when the weight of an edge is very small, the edge is barely there, so it should correspond to very high resistance. Having no edge corresponds to having a resistor of infinite resistance.

In this section, I use letters like a , b and c for the names of vertices, so I can reserve v for voltages.

Recall Ohm's law:

$$V = IR.$$

That is, the potential drop across a resistor (V) is equal to the current flowing over the resistor (I) times the resistance (R). To apply this in a graph, we will define for each edge (a, b) the current flowing *from u to a* to be $\mathbf{i}(a, b)$. As this is a directed quantity, we define

$$\mathbf{i}(b, a) = -\mathbf{i}(a, b).$$

I now let $\mathbf{v} \in \mathbb{R}^V$ be a vector of potentials (voltages) at vertices. Given these potentials, we can figure out how much current flows on each edge by the formula

$$\mathbf{i}(a, b) = \frac{1}{r_{a,b}} (\mathbf{v}(a) - \mathbf{v}(b)) = w_{a,b} (\mathbf{v}(a) - \mathbf{v}(b)).$$

That is, we adopt the convention that current flows from high voltage to low voltage. I would now like to write this equation in matrix form. The one complication is that each edge comes up twice in \mathbf{i} . So, to treat \mathbf{i} as a vector I will have each edge show up exactly once as (a, b) when $a < b$. I now define the *signed edge-vertex adjacency matrix* of the graph \mathbf{U} to be the matrix with rows indexed by edges and columns indexed by vertices such that

$$\mathbf{U}((a, b), c) = \begin{cases} 1 & \text{if } a = c \\ -1 & \text{if } b = c \\ 0 & \text{otherwise.} \end{cases}$$

In notation we defined earlier, we can write the row of \mathbf{U} corresponding to edge (a, b) as $\delta_a - \delta_b$. Define \mathbf{W} to be the diagonal matrix with rows and columns indexed by edges and the weights of edges on the diagonals. We then have

$$\mathbf{i} = \mathbf{W} \mathbf{U} \mathbf{v}.$$

Also recall that resistor networks cannot hold current. So, all the current entering a vertex a from edges in the graph must exit a to an external source. Let $\mathbf{i}_{ext} \in \mathbb{R}^V$ denote the external currents, where $\mathbf{i}_{ext}(a)$ is the amount of current entering the graph through node a . We then have

$$\mathbf{i}_{ext}(a) = \sum_{b:(a,b) \in E} \mathbf{i}(a, b).$$

In matrix form, this becomes

$$\mathbf{i}_{ext} = \mathbf{U}^T \mathbf{i} = \mathbf{U}^T \mathbf{W} \mathbf{U} \mathbf{v}. \quad (7.4)$$

The matrix

$$\mathbf{L} \stackrel{\text{def}}{=} \mathbf{U}^T \mathbf{W} \mathbf{U}$$

is, of course, the Laplacian. This is another way of writing the expression that we derived in Lecture 2:

$$\mathbf{L} = \sum_{(u,v) \in E} w_{u,v} (\delta_u - \delta_v) (\delta_u - \delta_v)^T.$$

It is often helpful to think of the nodes a for which $\mathbf{i}_{ext}(a) \neq 0$ as being *boundary* nodes. We will call the other nodes *internal*. Let's see what the equation

$$\mathbf{i}_{ext} = \mathbf{L} \mathbf{v}.$$

means for the internal nodes. If the graph is unweighted and a is an internal node, then the a th row of this equation is

$$0 = (\delta_a^T \mathbf{L}) \mathbf{v} = \sum_{(a,b) \in E} (\mathbf{v}(a) - \mathbf{v}(b)) = d(a) \mathbf{v}(a) - \sum_{(a,b) \in E} \mathbf{v}(b).$$

That is,

$$\mathbf{v}(a) = \frac{1}{d(a)} \sum_{(a,b) \in E} \mathbf{v}(b),$$

which means that the value of \mathbf{v} at a is the average of the values of \mathbf{v} at the neighbors of a . We find a similar relation in a weighted graph, except that it tells us that the value of \mathbf{v} at a is a weighted average of the values of \mathbf{v} at the neighbors of a .

7.6 Solving for currents

We are often interested in applying (7.4) in the reverse: given a vector of external currents \mathbf{i}_{ext} we solve for the induced voltages by

$$\mathbf{v} = \mathbf{L}^{-1} \mathbf{i}_{ext}.$$

This at first appears problematic, as the Laplacian matrix does not have an inverse. The way around this problem is to observe that we are only interested in solving these equations for vectors \mathbf{i}_{ext} for which the system has a solution. In the case of a connected graph, this equation will have a solution if the sum of the values of \mathbf{i}_{ext} is zero. That is, if the current going in to the circuit equals the current going out.

To obtain the solution to this equation, we multiply \mathbf{i}_{ext} by the Moore-Penrose *pseudo-inverse* of \mathbf{L} .

Definition 7.6.1. *The pseudo-inverse of a symmetric matrix \mathbf{L} , written \mathbf{L}^+ , is the matrix that has the same span as \mathbf{L} and that satisfies*

$$\mathbf{L}\mathbf{L}^+ = \mathbf{\Pi},$$

where $\mathbf{\Pi}$ is the symmetric projection onto the span of \mathbf{L} .

I remind you that a matrix $\mathbf{\Pi}$ is a symmetric projection if $\mathbf{\Pi}$ is symmetric and $\mathbf{\Pi}^2 = \mathbf{\Pi}$. This is equivalent to saying that all of its eigenvalues are 0 or 1.

The symmetric case is rather special. As $\mathbf{L}\mathbf{\Pi} = \mathbf{L}$, the other following properties of the Moore-Penrose pseudo inverse follow from this one:

$$\begin{aligned}\mathbf{L}^+\mathbf{L} &= \mathbf{\Pi}, \\ \mathbf{L}\mathbf{L}^+\mathbf{L} &= \mathbf{L} \\ \mathbf{L}^+\mathbf{L}\mathbf{L}^+ &= \mathbf{L}^+.\end{aligned}$$

It is easy to find a formula for the pseudo-inverse. First, let $\mathbf{\Psi}$ be the matrix whose i th column is ψ_i and let $\mathbf{\Lambda}$ be the diagonal matrix with λ_i on the i th diagonal. Recall that

$$\mathbf{L} = \mathbf{\Psi}\mathbf{\Lambda}\mathbf{\Psi}^T = \sum_i \lambda_i \psi_i \psi_i^T.$$

Claim 7.6.2.

$$\mathbf{L}^+ = \sum_{i>1} (1/\lambda_i) \psi_i \psi_i^T.$$

7.7 Random Walks

If there is time, I will do one more example.

A random walk on a graph is a process that occupies one vertex at each time step. We will consider walks that move to a random neighbor at each step, with probability proportional to the weight on the edge to the neighbor. So, if the walk is a vertex u , the chance that it moves to vertex v is

$$\frac{w_{u,v}}{d(u)}.$$

Now, distinguish two special nodes in the graph that we will call s and t . The walk will stop when it hits either s or t , so these are the boundary. Set $F = \{s, t\}$.

Let $\mathbf{x}(u)$ be the probability that a walk that starts at u will stop at s , rather than at t . We have the boundary conditions $\mathbf{x}(s) = 1$ and $\mathbf{x}(t) = 0$. For every other node u the chance that the walk stops at s is the sum over the neighbors v of u of the chance that the walk moves to v , times the chance that a walk from v stops at s . That is,

$$\mathbf{x}(u) = \sum_{v:(u,v) \in E} \frac{w_{u,v}}{d(u)} \mathbf{x}(v).$$

This are the same equations that voltages and spring networks satisfy!

As an example, let's take another look at the path graph P_n . Let's make $s = n$ and $t = 1$. So, the walk stops at either end. We then have $\mathbf{x}(n) = 1$, $\mathbf{x}(1) = 0$, and for every u in between,

$$\mathbf{x}(u) = \frac{u-1}{n-1}.$$

So, if the walk starts at node u , the chance that it ends at node n is $(u-1)/(n-1)$.

7.8 Exercise

Prove that for every $p > 0$

$$\mathbf{L}^p = \Psi \Lambda^p \Psi^T = \sum_i \lambda_i^p \psi_i \psi_i^T.$$

Moreover, this holds for any symmetric matrix. Not just Laplacians.