Effective Resistance and Schur Complements

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Disclaimer

These notes are not necessarily an accurate representation of what happened in class. The notes written before class say what I think I should say. I sometimes edit the notes after class to make them way what I wish I had said.

There may be small mistakes, so I recommend that you check any mathematically precise statement before using it in your own work.

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8.1 Overview

In this lecture, we will examine the behavior of a spring or resistor network on just a few vertices. We begin by considering how the whole behaves with respect to just two of the vertices, and show that it can be treated as one complex spring or resistor connecting the pair.

In general, we will see that if we wish to restrict our attention to a subset of the vertices, B, and if we require all other vertices to be internal, then we can construct a network just on B that factors out the contributions of the internal vertices. The process by which we do this is Gaussian elimination, and the Laplacian of the resulting network on B is called a Schur Complement.

8.2 Review

It will be convenient to define the family of *Laplacian Matrices.* I define Laplacian matrices to be the matrices that can arise as Laplacian matrices of graphs. That is, they are the matrices $\boldsymbol{L}$ that satisfy

1. $\boldsymbol{L}$ is symmetric,
2. all of the off-diagonal entries of $\boldsymbol{L}$ are non-positive, and
3. the row sums of $\boldsymbol{L}$ are zero, $\boldsymbol{L}\mathbf{1} = \mathbf{0}$.

Recall from last lecture that if $\boldsymbol{L}$ is the Laplacian matrix of a graph, viewed as a resistor network, and $\boldsymbol{v}$ are voltages as vertices, then the external currents are

$$\boldsymbol{i}_{ext} = \boldsymbol{L}\boldsymbol{v}.$$

For every vertex a,

$$\boldsymbol{i}_{ext}(a) = \sum_{b \sim a} \boldsymbol{i}(a,b) = \sum_{b \sim a} \frac{\boldsymbol{v}(a) - \boldsymbol{v}(b)}{r_{a,b}}.$$

We say that $\boldsymbol{v}$ is *harmonic* at a if

$$\boldsymbol{v}(a) = \frac{1}{d_a}\frac{\boldsymbol{v}(b)}{r_{a,b}} = \frac{1}{d_a} w_{a,b}\boldsymbol{v}(b) = \sum_{b \sim a} \boldsymbol{i}(a,b).$$

So, $\boldsymbol{v}$ is harmonic at a if the external current at a is zero.

We may reverse the relation between external current and voltage to obtain

$$\boldsymbol{v} = \boldsymbol{L}^{+}\boldsymbol{i}_{ext}.$$

Consider what this means when $\boldsymbol{i}_{ext}$ corresponds to a flow of one unit from vertex a to vertex b. The resulting voltages are

$$\boldsymbol{v} = \boldsymbol{L}^{+}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b).$$

Now, let c and d be two other vertices. The potential difference between c and d is

$$(\boldsymbol{\delta}_c - \boldsymbol{\delta}_d)^T \boldsymbol{v} = (\boldsymbol{\delta}_c - \boldsymbol{\delta}_d)^T \boldsymbol{L}^{+}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b).$$

Note the amazing reciprocity here: as $\boldsymbol{L}$ is symmetric this is equal to

$$(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b)^T \boldsymbol{L}^{+}(\boldsymbol{\delta}_c - \boldsymbol{\delta}_d).$$

So, the potential difference between c and d when we flow one unit from a to b is the same as the potential difference between a and b when we flow one unit from c to d.

8.3 Effective Resistance

The effective resistance between vertices a and b is the resistance between a and b given by the whole network. That is, if we treat the entire network as a resistor.

To figure out what this is, recall the equation

$$\boldsymbol{i}(a,b) = \frac{\boldsymbol{v}(a) - \boldsymbol{v}(b)}{r_{a,b}},$$

which holds for one resistor. We use the same equation to define the effective resistance of the whole network between a and b. That is, we consider an electrical flow that sends one unit of current into node a and removes one unit of current from node b. We then measure the potential difference between a and b that is required to realize this current, define this to be the effective resistance between a and b, and write it $\mathrm{R_{eff}}(a,b)$.

As we observed in the previous section, the potential difference between a and b in a flow of one unit of current from a to b is given by

$$(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b)^T \boldsymbol{L}^+ (\boldsymbol{\delta}_a - \boldsymbol{\delta}_b).$$

Effective resistance has many important properties. Some will appear in later lectures. Today, we will see that effective resistance is a distance. For now, I observe that it is the square of a Euclidean distance.

To this end, let $\boldsymbol{L}^{+/2}$ denote the square root of $\boldsymbol{L}^+$. Recall that every positive semidefinite matrix has a square root: the square root of a symmetric matrix $\boldsymbol{M}$ is the symmetric matrix $\boldsymbol{M}^{1/2}$ such that $(\boldsymbol{M}^{1/2})^2 = \boldsymbol{M}$. If

$$\boldsymbol{M} = \sum_i \lambda_i \boldsymbol{\psi}_i \boldsymbol{\psi}^T$$

is the spectral decomposition of $\boldsymbol{M}$, then

$$\boldsymbol{M}^{1/2} = \sum_i \lambda_i^{1/2} \boldsymbol{\psi}_i \boldsymbol{\psi}^T.$$

We now have

$$\begin{aligned}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b)^T \boldsymbol{L}^+ (\boldsymbol{\delta}_a - \boldsymbol{\delta}_b) &= \left(\boldsymbol{L}^{+/2}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b)\right)^T \boldsymbol{L}^{+/2}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b) = \left\| \boldsymbol{L}^{+/2}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b) \right\|^2 \\ &= \left\| \boldsymbol{L}^{+/2}\boldsymbol{\delta}_a - \boldsymbol{L}^{+/2}\boldsymbol{\delta}_b \right\|^2 = \mathrm{dist}(\boldsymbol{L}^{+/2}\boldsymbol{\delta}_a, \boldsymbol{L}^{+/2}\boldsymbol{\delta}_b)^2.\end{aligned}$$

8.4 The Effective Spring Constant

As you would imagine, we can also define the effective resistance through effective spring constants. In this case, we view the network of springs as one large compound network. If we define the spring constant to be the number w so that when a and b are stretched to distance l the potential energy in the spring is $wl^2/2$, then we should define the effective spring constant to be twice the entire energy of the network,

$$2\mathcal{E}(\boldsymbol{x}) \stackrel{\text{def}}{=} \sum_{(u,v)\in E} w_{u,v}(\boldsymbol{x}(u) - \boldsymbol{x}(v))^2,$$

when $\boldsymbol{x}(a)$ is fixed to 0 and $\boldsymbol{x}(b)$ is fixed to 1.

Fortunately, we already know how compute such a vector $\boldsymbol{x}$. Set

$$\boldsymbol{y} = \boldsymbol{L}^+(\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)/\mathrm{R_{eff}}(a,b).$$

We have

$$\boldsymbol{y}(b) - \boldsymbol{y}(a) = (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)^T \boldsymbol{L}^+ (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)/\mathrm{R_{eff}}(a,b) = 1,$$

and $\boldsymbol{y}$ is harmonic on $V - \{a,b\}$. So, we choose

$$\boldsymbol{x} = \boldsymbol{y} - \mathbf{1}\boldsymbol{y}(a).$$

The vector $\boldsymbol{x}$ satisfies $\boldsymbol{x}(a) = 0$, $\boldsymbol{x}(b) = 1$, and it is harmonic on $V - \{a,b\}$. So, it is the vector that minimizes the energy subject to the boundary conditions.

To finish, we compute the energy to be

$$\begin{aligned}
\boldsymbol{x}^T \boldsymbol{L} \boldsymbol{x} &= \boldsymbol{y}^T \boldsymbol{L} \boldsymbol{y} \\
&= \frac{1}{(\mathrm{R_{eff}}(a,b))^2} \left(\boldsymbol{L}^+(\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)\right)^T \boldsymbol{L} \left(\boldsymbol{L}^+(\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)\right) \\
&= \frac{1}{(\mathrm{R_{eff}}(a,b))^2} (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)^T \boldsymbol{L}^+ \boldsymbol{L} \boldsymbol{L}^+ (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a) \\
&= \frac{1}{(\mathrm{R_{eff}}(a,b))^2} (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)^T \boldsymbol{L}^+ (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a) \\
&= \frac{1}{\mathrm{R_{eff}}(a,b)}.
\end{aligned}$$

As the weights of edges are the reciprocals of their resistances, and the spring constant corresponds to the weight, this is the formula we would expect.

Resistor networks have an analogous quantity: the energy dissipation (into heat) when current flows through the network. It has the same formula. The reciprocal of the effective resistance is sometimes called the effective conductance.

8.5 An Example

In the case of a path graph with n vertices and edges of weight 1, the effective resistance between the extreme vertices is $n-1$.

In general, if a path consists of edges of resistance $r_{1,2}, \ldots, r_{n-1,n}$ then the effective resistance between the extreme vertices is

$$r_{1,2} + \cdots + r_{n-1,n}.$$

To see this, set the potential of vertex i to

$$\boldsymbol{v}(i) = r_{i,i+1} + \cdots + r_{n-1,n}.$$

Ohm's law then tells us that the current flow over the edge $(i, i+1)$ will be

$$\left(\boldsymbol{v}(i) - \boldsymbol{v}(i+1)\right)/r_{i,i+1} = 1.$$

If we have k parallel edges between two nodes s and t of resistances $r_1, \ldots, r_k$, then the effective resistance is

$$\mathrm{R_{eff}}(s,t) = \frac{1}{1/r_1 + \cdots + 1/r_k}.$$

Again, to see this, note that the flow over the ith edge will be

$$\frac{1/r_i}{1/r_1 + \cdots + 1/r_k},$$

so the total flow will be 1.

8.6 Equivalent Networks, Elimination, and Schur Complements

We have shown that the impact of the entire network on two vertices can be reduced to a network with one edge between them. We will now see that we can do the same for a subset of the vertices. I will do this in two ways: first by viewing $\boldsymbol{L}$ as an operator, and then by considering it as a quadratic form.

Let B be the subset of nodes that we would like to understand (B stands for *boundary*). All nodes not in B will be internal. Call them $I = V - B$.

As an operator, the Laplacian maps vectors of voltages to vectors of external currents. We want to examine what happens if we fix the voltages at vertices in B, and require the rest to be harmonic. Let $\boldsymbol{v}_B \in \mathbb{R}^B$ be the voltages at B. We want the matrix $\boldsymbol{L}_B$ such that

$$\boldsymbol{i}_B = \boldsymbol{L}_B \boldsymbol{v}_B$$

is the vector of external currents a vertices in B when we impose voltages $\boldsymbol{v}_B$ at vertices of B. As the internal vertices will have their voltages set to be harmonic, they will not have any external currents.

The remarkable fact that we will discover is that $\boldsymbol{L}_B$ is in fact a Laplacian matrix, and that it is obtained by performing Gaussian elimination to remove the internal vertices. **Warning:** $\boldsymbol{L}_B$ is not a submatrix of $\boldsymbol{L}$. To prove this, we will move from V to B by removing one vertex at a time. We'll start with a graph $G = (V, E, w)$, and we will set $B = \{2, \ldots, n\}$, and we will treat vertex 1 as internal.

We want to compute $\boldsymbol{L}\boldsymbol{v}$ given that $\boldsymbol{v}(b) = \boldsymbol{v}_B(b)$ for $b \in B$, and

$$\boldsymbol{v}(1) = \frac{1}{d_1} \sum_{b \sim 1} w_{1,b} \boldsymbol{v}_b(B).$$

That is, we want to substitute the value on the right-hand side for $\boldsymbol{v}(1)$ everywhere that it appears in the equation $\boldsymbol{i}_{ext} = \boldsymbol{L}\boldsymbol{v}$. The variable $\boldsymbol{v}(1)$ only appears in the equation for $\boldsymbol{i}_{ext}(a)$ when $(1, a) \in E$. When it does, it appears with coefficient $w_{1,a}$. Recall that the equation for $\boldsymbol{i}_{ext}(a)$ is

$$\boldsymbol{i}_{ext}(a) = d_a \boldsymbol{v}(a) - \sum_{b \sim a} w_{a,b} \boldsymbol{v}(b).$$

So, this substitution will result in removing $\boldsymbol{v}(1)$ from the equation and changing the equation for $\boldsymbol{i}_{ext}(a)$ to

$$\boldsymbol{i}_{ext}(a) = d_a \boldsymbol{v}(a) - \sum_{b \sim a, b \neq 1} w_{a,b} \boldsymbol{v}(b) - \frac{w_{1,a}}{d_1} \sum_{b \sim 1} w_{1,b} \boldsymbol{v}(b).$$

One of the terms counted in the right-most sum is in fact vertex a. So, we should write this as

$$\begin{aligned}\boldsymbol{i}_{ext}(a) = d_a \boldsymbol{v}(a) - \sum_{b \sim a, b \neq 1} w_{a,b} \boldsymbol{v}(b) - \frac{w_{1,a}}{d_1} \sum_{b \sim 1, b \neq a} w_{1,b} \boldsymbol{v}(b) - \frac{1}{d_1} w_{1,a}^2 \boldsymbol{v}(a) \\ = \left(d_a - \frac{w_{1,a}^2}{d_1} \right) \boldsymbol{v}(a) - \sum_{b \sim a, b \neq 1} w_{a,b} \boldsymbol{v}(b) - \frac{w_{1,a}}{d_1} \sum_{b \sim 1, b \neq a} w_{1,b} \boldsymbol{v}(b).\end{aligned}$$

This is how we eliminate $\boldsymbol{v}(1)$ from the equations. You should confirm that this is exactly what we do when we use Gaussian elimination to eliminate the variable $\boldsymbol{v}(1)$ by using the equality $\boldsymbol{i}_{ext}(1) = 0$.

We should now check that the matrix that encodes these equations is in fact a Laplacian matrix. To see that is is symmetric, observe that we have changed the entry in row a and column b by subtracting $w_{1,a} w_{1,b}$ from it. This is the same change that we make the entry in row b and column a. As we have subtracted a positive number, the off-diagonal entries remain non-positive. Finally, to confirm that the row-sums are 0, we could either note that we have subtracted a row of sum 0 from a row of sum 0, preserving sum 0, or by checking that

$$\sum_{b \sim a, b \neq 1} w_{a,b} + \frac{w_{1,a}}{d_1} \sum_{b \sim 1, b \neq a} w_{1,b} = d_a - w_{1,a} + \frac{w_{1,a}}{d_1}(d_1 - w_{1,a}) = d_a - \frac{w_{1,a}^2}{d_1}.$$

8.6.1 in terms of energy

I'm now going to try doing this in terms of the quadratic form. That is, we will compute the matrix $\boldsymbol{L}_B$ so that

$$\boldsymbol{v}_B^T \boldsymbol{L}_B \boldsymbol{v}_B = \boldsymbol{v}^T \boldsymbol{L} \boldsymbol{v},$$

given that $\boldsymbol{v}$ is harmonic at vertex 1 and agrees with $\boldsymbol{v}_B$ elsewhere. The quadratic form that we want to compute is thus given by

$$\begin{pmatrix} \frac{1}{d_1} \sum_{b \sim 1} w_{1,b} \boldsymbol{v}(b) \\ \boldsymbol{v}_B \end{pmatrix}^T \boldsymbol{L} \begin{pmatrix} \frac{1}{d_1} \sum_{b \sim 1} w_{1,b} \boldsymbol{v}(b) \\ \boldsymbol{v}_B \end{pmatrix}.$$

So that I can write this in terms of the entries of the Laplacian matrix, note that $d_1 = \boldsymbol{L}(1,1)$, and so

$$\boldsymbol{v}(1) = \frac{1}{d_1} \sum_{b \sim 1} w_{1,b} \boldsymbol{v}(b) = -(1/\boldsymbol{L}(1,1)) \boldsymbol{L}(1,B) \boldsymbol{v}_B.$$

Thus, we can write the quadratic form as

$$\begin{pmatrix} -(1/\boldsymbol{L}(1,1)) \boldsymbol{L}(1,B) \boldsymbol{v}_B \\ \boldsymbol{v}_B \end{pmatrix}^T \boldsymbol{L} \begin{pmatrix} -(1/\boldsymbol{L}(1,1)) \boldsymbol{L}(1,B) \boldsymbol{v}_B \\ \boldsymbol{v}_B \end{pmatrix}.$$

If we expand this out, we find that it equals

$$\begin{aligned}&\boldsymbol{v}_B^T \boldsymbol{L}(B,B) \boldsymbol{v}_B + \boldsymbol{L}(1,1) \left(-(1/\boldsymbol{L}(1,1)) \boldsymbol{L}(1,B) \boldsymbol{v}_B\right)^2 + 2\boldsymbol{v}(1) \boldsymbol{L}(1,B) \left(-(1/\boldsymbol{L}(1,1)) \boldsymbol{L}(1,B) \boldsymbol{v}_B\right) \\ &= \boldsymbol{v}_B^T \boldsymbol{L}(B,B) \boldsymbol{v}_B + \left(\boldsymbol{L}(1,B) \boldsymbol{v}_B\right)^2 / \boldsymbol{L}(1,1) - 2 \left(\boldsymbol{L}(1,B) \boldsymbol{v}_B\right)^2 / \boldsymbol{L}(1,1) \\ &= \boldsymbol{v}_B^T \boldsymbol{L}(B,B) \boldsymbol{v}_B - \left(\boldsymbol{L}(1,B) \boldsymbol{v}_B\right)^2 / \boldsymbol{L}(1,1).\end{aligned}$$

Thus,

$$\boldsymbol{L}_B = \boldsymbol{L}(B, B) - \frac{\boldsymbol{L}(B, 1)\boldsymbol{L}(1, B)}{\boldsymbol{L}(1, 1)}.$$

You should check that this is the matrix that appears in rows and columns 2 through n when we eliminate the entries in the first column of $\boldsymbol{L}$ by adding multiples of the first row.

We can again check that $\boldsymbol{L}_B$ is a Laplacian matrix. It is clear from the formula that it is symmetric and that the the off-diagonal entries are negative. To check that the constant vectors are in the nullspace, we can show that the quadratic form is zero on those vectors. If $\boldsymbol{v}_B$ is a constant vector, then $\boldsymbol{v}(1)$ must equal this constant, and so $\boldsymbol{v}$ is a constant vector and the value of the quadratic form is 0.

8.7 Eliminating Many Vertices

We can of course use the same procedure to eliminate many vertices. We begin by partitioning the vertex set into *boundary* vertices B and *internal* vertices I. We can then use Gaussian elimination to eliminate all of the internal vertices. You should recall that the submatrices produced by Gaussian elimination do not depend on the order of the eliminations. So, you may conclude that the matrix $\boldsymbol{L}_B$ is uniquely defined.

I will now give a nice formula for it that I mentioned in class, but did not have time to prove. Eliminating all of the vertices in I requires adding sums of rows of $\boldsymbol{L}$ in I to the rows not in I so as to eliminate all the entries of the columns in I. Let $b \in B$. The non-zero entries in the columns in I and row b are $\boldsymbol{L}(b, I)$. So, to eliminate those, we must subtract $\boldsymbol{L}(b, I)\boldsymbol{L}(I, I)^{-1}$ times the rows in I. That is, after the elimination, row b will become

$$\boldsymbol{L}(b, :) - \boldsymbol{L}(b, I)\boldsymbol{L}(I, I)^{-1}\boldsymbol{L}(I, :).$$

Putting this information together for every row in B, we see that after the eliminations the rows of the matrix indexed by B becomes

$$\boldsymbol{L}(B, :) - \boldsymbol{L}(B, I)\boldsymbol{L}(I, I)^{-1}\boldsymbol{L}(I, :).$$

If we restrict our attention to the columns that are also in B, we find the matrix that realizes the quadratic form in $\boldsymbol{v}_B$:

$$\boldsymbol{L}_B = \boldsymbol{L}(B, B) - \boldsymbol{L}(B, I)\boldsymbol{L}(I, I)^{-1}\boldsymbol{L}(I, B).$$

This is called the *Schur complement* of $\boldsymbol{L}$ with respect to the vertices in I. It is also the Laplacian of the graph we want on the vertices in B.

8.8 Effective Resistance is a Distance

A distance is any function on pairs of vertices such that

1. $\delta(a, a) = 0$ for every vertex a,
2. $\delta(a, b) \geq 0$ for all vertices a, b,
3. $\delta(a, b) = \delta(b, a)$, and
4. $\delta(a, c) \leq \delta(a, b) + \delta(b, c)$.

We claim that the effective resistance is a distance. The only non-trivial part to prove is the triangle inequality, (4).

Our proof will exploit a trivial but important observation about the voltages induced by a unit flow between a and b.

Claim 8.8.1. *Let $\boldsymbol{i}_{a,b}$ denote the vector that is 1 at a, -1 at b, and zero elsewhere. Let $\boldsymbol{v} = \boldsymbol{L}^{+}\boldsymbol{i}_{a,b}$. Then, for every vertex c,*

$$\boldsymbol{v}(a) \geq \boldsymbol{v}(c) \geq \boldsymbol{v}(b).$$

Proof. From the previous section, we know that there is a network just on $\{a, b, c\}$ that tells us where node c winds up when the potentials of a and b are fixed. So, the potential of c will be a weighted average of the potentials at a and b, which means it must lie between them. □

Lemma 8.8.2. *Let a, b and c be vertices in a graph. Then*

$$\mathrm{R_{eff}}(a, b) + \mathrm{R_{eff}}(b, c) \geq \mathrm{R_{eff}}(a, c).$$

Proof. Let $\boldsymbol{i}_{a,b}$ be the external current corresponding to sending one unit of current from a to b, and let $\boldsymbol{i}_{b,c}$ be the external current corresponding to sending one unit of current from b to c. Note that

$$\boldsymbol{i}_{a,c} = \boldsymbol{i}_{a,b} + \boldsymbol{i}_{b,c}.$$

Now, define the corresponding voltages by

$$\boldsymbol{v}_{a,b} = \boldsymbol{L}^{+}\boldsymbol{i}_{a,b} \quad \boldsymbol{v}_{b,c} = \boldsymbol{L}^{+}\boldsymbol{i}_{b,c}. \quad \boldsymbol{v}_{a,c} = \boldsymbol{L}^{+}\boldsymbol{i}_{a,c}.$$

By linearity, we have

$$\boldsymbol{v}_{a,c} = \boldsymbol{v}_{a,b} + \boldsymbol{v}_{b,c},$$

and so

$$\mathrm{R_{eff}}(a, c) = \boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{a,c} = \boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{a,b} + \boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{b,c}.$$

By Claim 8.8.1, we have

$$\boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{a,b} = \boldsymbol{v}_{a,b}(a) - \boldsymbol{v}_{a,b}(c) \leq \boldsymbol{v}_{a,b}(a) - \boldsymbol{v}_{a,b}(b) = \mathrm{R_{eff}}(a, b)$$

and similarly

$$\boldsymbol{i}_{a,c}^{T}\boldsymbol{v}_{b,c} \leq \mathrm{R_{eff}}(b, c).$$

The lemma follows. □

8.9 An interpretation of Gaussian elimination

This gives us a way of understand how Gaussian elimination solves a system of equations like $\boldsymbol{i}_{ext} = \boldsymbol{L}\boldsymbol{v}$. It constructs a sequence of graphs, $G_2, \ldots, G_n$, so that G_i is the effective network on vertices $i, \ldots, n$. It then solves for the entries of $\boldsymbol{v}$ backwards. Given $\boldsymbol{v}(i+1), \ldots, \boldsymbol{v}(n)$ and $\boldsymbol{i}_{ext}(i)$, we can solve for $\boldsymbol{v}(i)$. If $\boldsymbol{i}_{ext}(i) = 0$, then $\boldsymbol{v}(i)$ is set to the weighted average of its neighbors. If not, then we need to take $\boldsymbol{i}_{ext}(i)$ into account here and in the elimination as well. In the case in which we fix some vertices and let the rest be harmonic, there is no such complication.