

## Walks, Springs, and Resistor Networks

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## 12.1 Overview

In this lecture we will see how the analysis of random walks, spring networks, and resistor networks leads to the consideration of systems of linear equations in Laplacian matrices. The main purpose of this lecture is to introduce concepts and language that we will use extensively in the rest of the course.

## 12.2 Harmonic Functions

The theme of this whole lecture will be *harmonic functions* on graphs. These will be defined in terms of a weighted graph  $G = (V, E, w)$  and a set of *boundary* vertices  $B \subseteq V$ . We let  $S = V - B$  (I use “-” for set-minus). We will assume throughout this lecture that  $G$  is connected and that  $B$  is nonempty.

A function  $\mathbf{x} : V \rightarrow \mathbf{R}$  is said to be *harmonic* at a vertex  $a$  if the value of  $\mathbf{x}$  at  $a$  is the weighted average of its values at the neighbors of  $a$  where the weights are given by  $w$ :

$$\mathbf{x}(a) = \frac{1}{d_a} \sum_{b \sim a} w_{a,b} \mathbf{x}(b). \quad (12.1)$$

The function  $\mathbf{x}$  is harmonic on  $S$  if it is harmonic for all  $a \in S$ .

## 12.3 Random Walks on Graphs

Consider the standard (not lazy) random walk on the graph  $G$ . Recall that when the walk is at a vertex  $a$ , the probability it moves to a neighbor  $b$  is

$$\frac{w_{a,b}}{d_a}.$$

Distinguish two special nodes in the graph that we will call  $s$  and  $t$ , and run the random walk until it hits either  $s$  or  $t$ . We view  $s$  and  $t$  as the boundary, so  $B = \{s, t\}$ .

Let  $\mathbf{x}(a)$  be the probability that a walk that starts at  $a$  will stop at  $s$ , rather than at  $t$ . We have the boundary conditions  $\mathbf{x}(s) = 1$  and  $\mathbf{x}(t) = 0$ . For every other node  $a$  the chance that the walk

stops at  $s$  is the sum over the neighbors  $b$  of  $a$  of the chance that the walk moves to  $b$ , times the chance that a walk from  $b$  stops at  $s$ . That is,

$$\mathbf{x}(a) = \sum_{b \sim a} \frac{w_{a,b}}{d_a} \mathbf{x}(b).$$

So, the function  $\mathbf{x}$  is harmonic at every vertex in  $V - B$ .

For example, consider the path graph  $P_n$ . Let's make  $s = n$  and  $t = 1$ . So, the walk stops at either end. We then have  $\mathbf{x}(n) = 1$ ,  $\mathbf{x}(1) = 0$ . It is easy to construction at least one solution to the harmonic equations (12.1): we can set

$$\mathbf{x}(a) = \frac{a - 1}{n - 1}.$$

It essentially follows from the definitions that there can be only one vector  $\mathbf{x}$  that solves these equations. But, we will prove this algebraically later in lecture.

These solutions tell us that if the walk starts at node  $a$ , the chance that it ends at node  $n$  is  $(a - 1)/(n - 1)$ . This justifies some of our analysis of the Bolas graph from Lecture 10.

Of course, the exact same analysis goes through for the lazy random walks: those give

$$\mathbf{x}(a) = (1/2)\mathbf{x}(a) + (1/2) \sum_{b \sim a} \frac{w_{a,b}}{d_a} \mathbf{x}(b) \iff \mathbf{x}(a) = \sum_{b \sim a} \frac{w_{a,b}}{d_a} \mathbf{x}(b).$$

## 12.4 Spring Networks

We begin by imagining that every edge of a graph  $G = (V, E)$  is an ideal spring or rubber band. They are joined together at the vertices. Given such a structure, we will pick a subset of the vertices  $B \subseteq V$  and fix the location of every vertex in  $B$ . For example, you could nail each vertex in  $B$  onto a point in the real line, or onto a board in  $\mathbb{R}^2$ . We will then study where the other vertices wind up.

We can use Hooke's law to figure this out. To begin, assume that each rubber band is an ideal spring with spring constant 1. If your graph is weighted, then the spring constant of each edge should be its weight. If a rubber band connects vertices  $a$  and  $b$ , then Hooke's law tells us that the force it exerts at node  $a$  is in the direction of  $b$  and is proportional to the distance between  $a$  and  $b$ . Let  $\mathbf{x}(a)$  be the position of each vertex  $a$ . You should begin by thinking of  $\mathbf{x}(a)$  being in  $\mathbb{R}$ , but you will see that it is just as easy to make it a vector in  $\mathbb{R}^2$  or  $\mathbb{R}^k$  for any  $k$ .

The force the rubber band between  $a$  and  $b$  exerts on  $a$  is

$$\mathbf{x}(b) - \mathbf{x}(a).$$

In a stable configuration, all of the vertices that have not been nailed down must experience a zero

net force. That is

$$\begin{aligned} \sum_{b \sim a} (\mathbf{x}(b) - \mathbf{x}(a)) = 0 &\iff \sum_{b \sim a} \mathbf{x}(b) = d_a \mathbf{x}(a) \\ &\iff \frac{1}{d_a} \sum_{b \sim a} \mathbf{x}(b) = \mathbf{x}(a). \end{aligned}$$

In a stable configuration, every vertex that is not on the boundary must be the average of its neighbors.

In the weighted case, we would have for each  $a \in V - B$

$$\frac{1}{d_a} \sum_{b \sim a} w_{a,b} \mathbf{x}(b) = \mathbf{x}(a).$$

That is,  $\mathbf{x}$  is *harmonic* on  $V - B$ .

We will next show that the equations (12.1) have a solution, and that it is unique<sup>1</sup> if the underlying graph is connected and  $B$  is nonempty. But first, consider again the path graph  $P_n$  with the endpoints fixed:  $B = \{1, n\}$ . Let us fix them to the values  $\mathbf{f}(1) = 1$  and  $\mathbf{f}(n) = n$ . The only solution to the equations (12.1) is the obvious one: vertex  $i$  is mapped to  $i$ :  $\mathbf{x}(i) = i$  for all  $i$ .

## 12.5 Laplacian linear equations

If we rewrite equation (12.1) as

$$d_a \mathbf{x}(a) - \sum_{b \sim a} w_{a,b} \mathbf{x}(b) = 0, \tag{12.2}$$

we see that it corresponds to the row of the Laplacian matrix corresponding to vertex  $a$ . So, we may find a solution to the equations (12.1) by solving a system of equations in the submatrix of the Laplacian indexed by vertices in  $V - B$ .

To be more concrete, I will set up those equations. For each vertex  $a \in B$ , let its position be fixed to  $\mathbf{f}(a)$ . Then, we can re-write equation (12.2) as

$$d_a \mathbf{x}(a) - \sum_{b \notin B: (a,b) \in E} w_{a,b} \mathbf{x}(b) = \sum_{b \in B: (a,b) \in E} w_{a,b} \mathbf{f}(b),$$

for each  $a \in V - B$ . So, all of the boundary terms wind up in the right-hand vector.

Let  $S = V - B$ . We now see that this is an equation of the form

$$\mathbf{L}(S, S) \mathbf{x}(S) = \mathbf{r}, \quad \text{with } \mathbf{r} = \mathbf{M}(S, :) \mathbf{f}.$$

By  $\mathbf{L}(S, S)$  I mean the submatrix of  $\mathbf{L}$  indexed by rows and columns of  $S$ , and by  $\mathbf{x}(S)$  I mean the sub-vector of  $\mathbf{x}$  indexed by  $S$ .

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<sup>1</sup>It can only fail to be unique if there is a connected component that contains no vertices of  $B$ .

We can then write the condition that entries of  $B$  are fixed to  $\mathbf{f}$  by

$$\mathbf{x}(B) = \mathbf{f}(B).$$

We have reduced the problem to that of solving a system of equations in a submatrix of the Laplacian.

Submatrices of Laplacians are a lot like Laplacians, except that they are positive definite. To see this, note that all of the off-diagonals of the submatrix of  $\mathbf{L}$  agree with all the off-diagonals of the Laplacian of the induced subgraph on the internal vertices. But, some of the diagonals are larger: the diagonals of nodes in the submatrix account for both edges in the induced subgraph and edges to the vertices in  $B$ .

**Claim 12.5.1.** *Let  $\mathbf{L}$  be the Laplacian of  $G = (V, E, w)$ , let  $B \subseteq V$ , and let  $S = V - B$ . Then,*

$$\mathbf{L}(S, S) = \mathbf{L}_{G(S)} + \mathbf{X}_S,$$

where  $G(S)$  is the subgraph induced on the vertices in  $S$  and  $\mathbf{X}_S$  is the diagonal matrix with entries

$$\mathbf{X}_S(a, a) = \sum_{b \sim a, b \in B} w_{a,b}, \quad \text{for } a \in S.$$

**Lemma 12.5.2.** *Let  $\mathbf{L}$  be the Laplacian matrix of a connected graph and let  $\mathbf{X}$  be a nonnegative, diagonal matrix with at least one nonzero entry. Then,  $\mathbf{L} + \mathbf{X}$  is positive definite.*

*Proof.* We will prove that  $\mathbf{x}^T(\mathbf{L} + \mathbf{X})\mathbf{x} > 0$  for every nonzero vector  $\mathbf{x}$ . As both  $\mathbf{L}$  and  $\mathbf{X}$  are positive semidefinite, we have

$$\mathbf{x}^T(\mathbf{L} + \mathbf{X})\mathbf{x} \geq \min(\mathbf{x}^T \mathbf{L} \mathbf{x}, \mathbf{x}^T \mathbf{X} \mathbf{x}).$$

As the graph is connected,  $\mathbf{x}^T \mathbf{L} \mathbf{x}$  is positive unless  $\mathbf{x}$  is a constant vector. If  $\mathbf{x} = c\mathbf{1}$  for some  $c \neq 0$ , then we obtain

$$c^2 \mathbf{1}^T (\mathbf{L} + \mathbf{X}) \mathbf{1} = c^2 \mathbf{1}^T \mathbf{X} \mathbf{1} = c^2 \sum_i \mathbf{X}(i, i) > 0.$$

□

**Lemma 12.5.3.** *Let  $\mathbf{L}$  be the Laplacian matrix of a connected graph  $G = (V, E, w)$ , let  $B$  be a nonempty, proper subset of  $V$ , and let  $S = V - B$ . Then,  $\mathbf{L}(S, S)$  is positive definite.*

*Proof.* Let  $S_1, \dots, S_k$  be the connected components of vertices of  $G(S)$ . We can use these to write  $\mathbf{L}(S, S)$  as a block matrix with blocks equal to  $\mathbf{L}(S_i, S_i)$ . Each of these blocks can be written

$$\mathbf{L}(S_i, S_i) = \mathbf{L}_{G_{S_i}} + \mathbf{X}_{S_i}.$$

As  $G$  is connected, there must be some vertex in  $S_i$  with an edge to a vertex not in  $S_i$ . This implies that  $\mathbf{X}_{S_i}$  is not the zero matrix, and so we can apply Lemma 12.5.2 to prove that  $\mathbf{L}(S_i, S_i)$  is invertible. □

As the matrix  $\mathbf{L}(S, S)$  is invertible, the equations have a solution, and it must be unique.

## 12.6 Energy

Physics also tells us that the vertices will settle into the position that minimizes the potential energy. The potential energy of an ideal linear spring with constant  $w$  when stretched to length  $l$  is

$$\frac{1}{2}wl^2.$$

So, the potential energy in a configuration  $\mathbf{x}$  is given by

$$\mathcal{E}(\mathbf{x}) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{(a,b) \in E} w_{a,b}(\mathbf{x}(a) - \mathbf{x}(b))^2. \quad (12.3)$$

For any  $\mathbf{x}$  that minimizes the energy, the partial derivative of the energy with respect to each variable must be zero. In this case, the variables are  $\mathbf{x}(a)$  for  $a \in S$ . The partial derivative with respect to  $\mathbf{x}(a)$  is

$$\frac{1}{2} \sum_{b \sim a} w_{a,b} 2(\mathbf{x}(a) - \mathbf{x}(b)) = \sum_{b \sim a} w_{a,b}(\mathbf{x}(a) - \mathbf{x}(b)).$$

Setting this to zero gives the equations we previously derived: (12.1).

## 12.7 Resistor Networks

We now consider a related physical model of a graph in which we treat every edge as a resistor. If the graph is unweighted, we will assume that each resistor has resistance 1. If an edge  $e$  has weight  $w_e$ , we will give the corresponding resistor resistance  $r_e = 1/w_e$ . The reason is that when the weight of an edge is very small, the edge is barely there, so it should correspond to very high resistance. Having no edge corresponds to having a resistor of infinite resistance.

Recall Ohm's law:

$$V = IR.$$

That is, the potential drop across a resistor ( $V$ ) is equal to the current flowing over the resistor ( $I$ ) times the resistance ( $R$ ). To apply this in a graph, we will define for each edge  $(a, b)$  the current flowing *from*  $a$  *to*  $b$  to be  $\mathbf{i}(a, b)$ . As this is a directed quantity, we define

$$\mathbf{i}(b, a) = -\mathbf{i}(a, b).$$

I now let  $\mathbf{v} \in \mathbb{R}^V$  be a vector of potentials (voltages) at vertices. Given these potentials, we can figure out how much current flows on each edge by the formula

$$\mathbf{i}(a, b) = \frac{1}{r_{a,b}} (\mathbf{v}(a) - \mathbf{v}(b)) = w_{a,b} (\mathbf{v}(a) - \mathbf{v}(b)).$$

That is, we adopt the convention that current flows from high voltage to low voltage. We would like to write this equation in matrix form. The one complication is that each edge comes up twice in  $\mathbf{i}$ . So, to treat  $\mathbf{i}$  as a vector we will have each edge show up exactly once as  $(a, b)$  when  $a < b$ .

We now define the *signed edge-vertex adjacency matrix* of the graph  $\mathbf{U}$  to be the matrix with rows indexed by edges and columns indexed by vertices such that

$$\mathbf{U}((a, b), c) = \begin{cases} 1 & \text{if } a = c \\ -1 & \text{if } b = c \\ 0 & \text{otherwise.} \end{cases}$$

Thus the row of  $\mathbf{U}$  corresponding to edge  $(a, b)$  is  $\mathbf{U}((a, b), :) = \boldsymbol{\delta}_a^T - \boldsymbol{\delta}_b^T$ .

Define  $\mathbf{W}$  to be the diagonal matrix with rows and columns indexed by edges with the weights of the edges on the diagonals. We then have

$$\mathbf{i} = \mathbf{W} \mathbf{U} \mathbf{v}.$$

Also recall that resistor networks cannot hold current. So, all the current entering a vertex  $a$  from edges in the graph must exit  $a$  to an external source. Let  $\mathbf{i}_{ext} \in \mathbb{R}^V$  denote the external currents, where  $\mathbf{i}_{ext}(a)$  is the amount of current entering the graph through node  $a$ . We then have

$$\mathbf{i}_{ext}(a) = \sum_{b \sim a} \mathbf{i}(a, b).$$

In matrix form, this becomes

$$\mathbf{i}_{ext} = \mathbf{U}^T \mathbf{i} = \mathbf{U}^T \mathbf{W} \mathbf{U} \mathbf{v}. \quad (12.4)$$

The matrix

$$\mathbf{L} \stackrel{\text{def}}{=} \mathbf{U}^T \mathbf{W} \mathbf{U}$$

is, of course, the Laplacian. This is another way of writing the expression that we derived in Lecture 3:

$$\mathbf{L} = \sum_{a \sim b} w_{a,b} (\boldsymbol{\delta}_a - \boldsymbol{\delta}_b) (\boldsymbol{\delta}_a - \boldsymbol{\delta}_b)^T.$$

It is often helpful to think of the nodes  $a$  for which  $\mathbf{i}_{ext}(a) \neq 0$  as being *boundary* nodes. We will call the other nodes *internal*. Let's see what the equation

$$\mathbf{i}_{ext} = \mathbf{L} \mathbf{v}.$$

means for the internal nodes. If the graph is unweighted and  $a$  is an internal node, then the  $a$ th row of this equation is

$$0 = (\boldsymbol{\delta}_a^T \mathbf{L}) \mathbf{v} = \sum_{a \sim b} (\mathbf{v}(a) - \mathbf{v}(b)) = d_a \mathbf{v}(a) - \sum_{a \sim b} \mathbf{v}(b).$$

That is,

$$\mathbf{v}(a) = \frac{1}{d_a} \sum_{a \sim b} \mathbf{v}(b),$$

which means that  $\mathbf{v}$  is harmonic at  $a$ . Of course, the same holds in weighted graphs.

## 12.8 Solving for currents

We are often interested in applying (12.4) in the reverse: given a vector of external currents  $\mathbf{i}_{ext}$  we solve for the induced voltages by

$$\mathbf{v} = \mathbf{L}^{-1}\mathbf{i}_{ext}.$$

This at first appears problematic, as the Laplacian matrix does not have an inverse. The way around this problem is to observe that we are only interested in solving these equations for vectors  $\mathbf{i}_{ext}$  for which the system has a solution. In the case of a connected graph, this equation will have a solution if the sum of the values of  $\mathbf{i}_{ext}$  is zero. That is, if the current going in to the circuit equals the current going out. These are precisely the vectors that are in the span of the Laplacian.

To obtain the solution to this equation, we multiply  $\mathbf{i}_{ext}$  by the Moore-Penrose *pseudo-inverse* of  $\mathbf{L}$ .

**Definition 12.8.1.** *The pseudo-inverse of a symmetric matrix  $\mathbf{L}$ , written  $\mathbf{L}^+$ , is the matrix that has the same span as  $\mathbf{L}$  and that satisfies*

$$\mathbf{L}\mathbf{L}^+ = \mathbf{\Pi},$$

where  $\mathbf{\Pi}$  is the symmetric projection onto the span of  $\mathbf{L}$ .

I remind you that a matrix  $\mathbf{\Pi}$  is a symmetric projection if  $\mathbf{\Pi}$  is symmetric and  $\mathbf{\Pi}^2 = \mathbf{\Pi}$ . This is equivalent to saying that all of its eigenvalues are 0 or 1. We also know that  $\mathbf{\Pi} = (1/n)\mathbf{L}_{K_n}$ .

The symmetric case is rather special. As  $\mathbf{L}\mathbf{\Pi} = \mathbf{L}$ , the other following properties of the Moore-Penrose pseudo inverse follow from this one:

$$\begin{aligned}\mathbf{L}^+\mathbf{L} &= \mathbf{\Pi}, \\ \mathbf{L}\mathbf{L}^+\mathbf{L} &= \mathbf{L} \\ \mathbf{L}^+\mathbf{L}\mathbf{L}^+ &= \mathbf{L}^+.\end{aligned}$$

It is easy to find a formula for the pseudo-inverse. First, let  $\mathbf{\Psi}$  be the matrix whose  $i$ th column is  $\psi_i$  and let  $\mathbf{A}$  be the diagonal matrix with  $\lambda_i$  on the  $i$ th diagonal. Recall that

$$\mathbf{L} = \mathbf{\Psi}\mathbf{A}\mathbf{\Psi}^T = \sum_i \lambda_i \psi_i \psi_i^T.$$

**Claim 12.8.2.**

$$\mathbf{L}^+ = \sum_{i>1} (1/\lambda_i) \psi_i \psi_i^T.$$

## 12.9 Electrical Flows and Effective Resistance

We now know that if a resistor network has external currents  $\mathbf{i}_{ext}$ , then the voltages induced at the vertices will be given by

$$\mathbf{v} = \mathbf{L}^+\mathbf{i}_{ext}.$$

Consider what this means when  $\mathbf{i}_{ext}$  corresponds to a flow of one unit from vertex  $a$  to vertex  $b$ . The resulting voltages are

$$\mathbf{v} = \mathbf{L}^+(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b).$$

Now, let  $c$  and  $d$  be two other vertices. The potential difference between  $c$  and  $d$  is

$$\mathbf{v}(c) - \mathbf{v}(d) = (\boldsymbol{\delta}_c - \boldsymbol{\delta}_d)^T \mathbf{v} = (\boldsymbol{\delta}_c - \boldsymbol{\delta}_d)^T \mathbf{L}^+(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b).$$

Note the amazing reciprocity here: as  $\mathbf{L}$  is symmetric this is equal to

$$(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b)^T \mathbf{L}^+(\boldsymbol{\delta}_c - \boldsymbol{\delta}_d).$$

So, the potential difference between  $c$  and  $d$  when we flow one unit from  $a$  to  $b$  is the same as the potential difference between  $a$  and  $b$  when we flow one unit from  $c$  to  $d$ .

The *effective resistance* between vertices  $a$  and  $b$  is the resistance between  $a$  and  $b$  when we view the entire network as one complex resistor.

To figure out what this is, recall the equation

$$\mathbf{i}(a, b) = \frac{\mathbf{v}(a) - \mathbf{v}(b)}{r_{a,b}},$$

which holds for one resistor. We use the same equation to define the effective resistance of the whole network between  $a$  and  $b$ . That is, we consider an electrical flow that sends one unit of current into node  $a$  and removes one unit of current from node  $b$ . We then measure the potential difference between  $a$  and  $b$  that is required to realize this current, define this to be the effective resistance between  $a$  and  $b$ , and write it  $R_{\text{eff}}(a, b)$ . As it equals the potential difference between  $a$  and  $b$  in a flow of one unit of current from  $a$  to  $b$ :

$$R_{\text{eff}}(a, b) = (\boldsymbol{\delta}_a - \boldsymbol{\delta}_b)^T \mathbf{L}^+(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b).$$

## 12.10 Exercise

Prove that for every  $p > 0$

$$\mathbf{L}^p = \boldsymbol{\Psi} \boldsymbol{\Lambda}^p \boldsymbol{\Psi}^T = \sum_i \lambda_i^p \boldsymbol{\psi}_i \boldsymbol{\psi}_i^T.$$

Moreover, this holds for any symmetric matrix. Not just Laplacians.

## References