Spectral Graph Theory Lecture 13

Effective Resistance and Schur Complements

Daniel A. Spielman October 10, 2018

13.1 Introduction

In the last lecture, we encountered two types of physical problems on graphs. The first were problems with fixed boundary conditions. In these, we are given fixed values of $\boldsymbol{x}(b)$ for some nodes $b \in B$, and must either find the values of $\boldsymbol{x}(a)$ for $a \in S = V - B$ that either minimize the Laplacian quadratic form (energy) or so that the vector $\boldsymbol{x}$ is harmonic at all of these a. These are the same problems.

We learned that the solution is given by

$$\boldsymbol{x}(S) = \boldsymbol{L}(S,S)^{-1}\boldsymbol{M}(S,B)\boldsymbol{x}(B).$$

As $\boldsymbol{M}$ is the adjacency matrix and S and B are disjoint, we have $\boldsymbol{M}(S,B) = -\boldsymbol{L}(S,B)$, giving the formula

$$\boldsymbol{x}(S) = -\boldsymbol{L}(S,S)^{-1}\boldsymbol{L}(S,B)\boldsymbol{x}(B).$$

The other problem we saw was that of computing the voltages that are induced by fixing certain external flows. These were solved by the equations

$$\boldsymbol{v} = \boldsymbol{L}^{+}\boldsymbol{i}_{ext}.$$

For those vertices a for which $\boldsymbol{i}_{ext}(a) = 0$, this equation will result in $\boldsymbol{v}$ being harmonic at a. The previous problem corresponds to fixing voltages at some vertices, rather than fixing flows.

We then defined the *effective* resistance between vertices a and b to be the potential difference between a and b in the unit flow of one unit from a to b:

$$\mathrm{R_{eff}}(a,b) = (\boldsymbol{\delta}_a - \boldsymbol{\delta}_b)^T \boldsymbol{L}^{+}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b).$$

That is, this is the resistance between a and b imposed by the network as a whole.

An alternative way of saying that is that if we only care about vertices a and b, we can reduce the rest of the network to a single edge.

In general, we will see that if we wish to restrict our attention to a subset of the vertices, B, and if we require all other vertices to be internal, then we can construct a network just on B that factors out the contributions of the internal vertices. The process by which we do this is Gaussian elimination, and the Laplacian of the resulting network on B is called a Schur Complement.

We will also show that effective resistance is a distance. Other important properties of effective resistance will appear in later lectures.

For now, I observe that effective resistance is the square of a Euclidean distance.

To this end, let $\boldsymbol{L}^{+/2}$ denote the square root of $\boldsymbol{L}^{+}$. Recall that every positive semidefinite matrix has a square root: the square root of a symmetric matrix $\boldsymbol{M}$ is the symmetric matrix $\boldsymbol{M}^{1/2}$ such that $(\boldsymbol{M}^{1/2})^2 = \boldsymbol{M}$. If

$$\boldsymbol{M} = \sum_i \lambda_i \boldsymbol{\psi}_i \boldsymbol{\psi}^T$$

is the spectral decomposition of $\boldsymbol{M}$, then

$$\boldsymbol{M}^{1/2} = \sum_i \lambda_i^{1/2} \boldsymbol{\psi}_i \boldsymbol{\psi}^T.$$

We now have

$$\begin{aligned}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b)^T \boldsymbol{L}^{+} (\boldsymbol{\delta}_a - \boldsymbol{\delta}_b) = \left(\boldsymbol{L}^{+/2}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b)\right)^T \boldsymbol{L}^{+/2}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b) &= \left\|\boldsymbol{L}^{+/2}(\boldsymbol{\delta}_a - \boldsymbol{\delta}_b)\right\|^2 \\ &= \left\|\boldsymbol{L}^{+/2}\boldsymbol{\delta}_a - \boldsymbol{L}^{+/2}\boldsymbol{\delta}_b\right\|^2 = \operatorname{dist}(\boldsymbol{L}^{+/2}\boldsymbol{\delta}_a, \boldsymbol{L}^{+/2}\boldsymbol{\delta}_b)^2.\end{aligned}$$

13.2 Effective Resistance through Energy Minimization

As you would imagine, we can also define the effective resistance through effective spring constants. In this case, we view the network of springs as one large compound network. If we define the spring constant to be the number w so that when a and b are stretched to distance l the potential energy in the spring is $wl^2/2$, then we should define the effective spring constant to be twice the entire energy of the network,

$$2\mathcal{E}\left(\boldsymbol{x}\right) \stackrel{\text{def}}{=} \sum_{(u,v)\in E} w_{u,v}(\boldsymbol{x}(u) - \boldsymbol{x}(v))^2,$$

when $\boldsymbol{x}(a)$ is fixed to 0 and $\boldsymbol{x}(b)$ is fixed to 1.

Fortunately, we already know how compute such a vector $\boldsymbol{x}$. Set

$$\boldsymbol{y} = \boldsymbol{L}^{+}(\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)/\mathrm{R_{eff}}(a,b).$$

We have

$$\boldsymbol{y}(b) - \boldsymbol{y}(a) = (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)^T \boldsymbol{L}^{+}(\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)/\mathrm{R_{eff}}(a,b) = 1,$$

and $\boldsymbol{y}$ is harmonic on $V - \{a, b\}$. So, we choose

$$\boldsymbol{x} = \boldsymbol{y} - \mathbf{1}\boldsymbol{y}(a).$$

The vector $\boldsymbol{x}$ satisfies $\boldsymbol{x}(a) = 0$, $\boldsymbol{x}(b) = 1$, and it is harmonic on $V - \{a, b\}$. So, it is the vector that minimizes the energy subject to the boundary conditions.

To finish, we compute the energy to be

$$\begin{aligned}
\boldsymbol{x}^T \boldsymbol{L} \boldsymbol{x} &= \boldsymbol{y}^T \boldsymbol{L} \boldsymbol{y} \\
&= \frac{1}{(\mathrm{R_{eff}}(a,b))^2} \left(\boldsymbol{L}^+ (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a) \right)^T \boldsymbol{L} \left(\boldsymbol{L}^+ (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a) \right) \\
&= \frac{1}{(\mathrm{R_{eff}}(a,b))^2} (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)^T \boldsymbol{L}^+ \boldsymbol{L} \boldsymbol{L}^+ (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a) \\
&= \frac{1}{(\mathrm{R_{eff}}(a,b))^2} (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a)^T \boldsymbol{L}^+ (\boldsymbol{\delta}_b - \boldsymbol{\delta}_a) \\
&= \frac{1}{\mathrm{R_{eff}}(a,b)}.
\end{aligned}$$

As the weights of edges are the reciprocals of their resistances, and the spring constant corresponds to the weight, this is the formula we would expect.

Resistor networks have an analogous quantity: the energy dissipation (into heat) when current flows through the network. It has the same formula. The reciprocal of the effective resistance is sometimes called the effective conductance.

13.3 Examples: Series and Parallel

In the case of a path graph with n vertices and edges of weight 1, the effective resistance between the extreme vertices is $n-1$.

In general, if a path consists of edges of resistance $r_{1,2}, \ldots, r_{n-1,n}$ then the effective resistance between the extreme vertices is

$$r_{1,2} + \cdots + r_{n-1,n}.$$

To see this, set the potential of vertex i to

$$\boldsymbol{v}(i) = r_{i,i+1} + \cdots + r_{n-1,n}.$$

Ohm's law then tells us that the current flow over the edge $(i, i+1)$ will be

$$\left(\boldsymbol{v}(i) - \boldsymbol{v}(i+1) \right) / r_{i,i+1} = 1.$$

If we have k parallel edges between two nodes s and t of resistances $r_1, \ldots, r_k$, then the effective resistance is

$$\mathrm{R_{eff}}(s,t) = \frac{1}{1/r_1 + \cdots + 1/r_k}.$$

To see this, impose a potential difference of 1 between s and t. This will induce a flow of $1/r_i = w_i$ on edge i. So, the total flow will be

$$\sum_i = 1/r_i = \sum_i w_i.$$

13.4 Equivalent Networks, Elimination, and Schur Complements

We have shown that the impact of the entire network on two vertices can be reduced to a network with one edge between them. We will now see that we can do the same for a subset of the vertices. I will do this in two ways: first by viewing $\boldsymbol{L}$ as an operator, and then by considering it as a quadratic form.

Let B be the subset of nodes that we would like to understand (B stands for *boundary*). All nodes not in B will be internal. Call them $I = V - B$.

As an operator, the Laplacian maps vectors of voltages to vectors of external currents. We want to examine what happens if we fix the voltages at vertices in B, and require the rest to be harmonic. Let $\boldsymbol{v}(B) \in \mathbb{R}^B$ be the voltages at B. We want the matrix $\boldsymbol{L}_B$ such that

$$\boldsymbol{i}_B = \boldsymbol{L}_B \boldsymbol{v}(B)$$

is the vector of external currents a vertices in B when we impose voltages $\boldsymbol{v}(B)$ at vertices of B. As the internal vertices will have their voltages set to be harmonic, they will not have any external currents.

The remarkable fact that we will discover is that $\boldsymbol{L}_B$ is in fact a Laplacian matrix, and that it is obtained by performing Gaussian elimination to remove the internal vertices. **Warning:** $\boldsymbol{L}_B$ is not a submatrix of $\boldsymbol{L}$. To prove this, we will move from V to B by removing one vertex at a time. We'll start with a graph $G = (V, E, w)$, and we will set $B = \{2, \ldots, n\}$, and we will treat vertex 1 as internal. Let N denote the set of neighbors of vertex 1.

We want to compute $\boldsymbol{L}\boldsymbol{v}$ given that $\boldsymbol{v}(b) = \boldsymbol{v}(B)(b)$ for $b \in B$, and

$$\boldsymbol{v}(1) = \frac{1}{d_1} \sum_{a \in N} w_{1,a} \boldsymbol{v}(a). \tag{13.1}$$

That is, we want to substitute the value on the right-hand side for $\boldsymbol{v}(1)$ everywhere that it appears in the equation $\boldsymbol{i}_{ext} = \boldsymbol{L}\boldsymbol{v}$. The variable $\boldsymbol{v}(1)$ only appears in the equation for $\boldsymbol{i}_{ext}(a)$ when $a \in N$. When it does, it appears with coefficient $w_{1,a}$. Recall that the equation for $\boldsymbol{i}_{ext}(b)$ is

$$\boldsymbol{i}_{ext}(b) = d_b \boldsymbol{v}(b) - \sum_{c \sim b} w_{b,c} \boldsymbol{v}(c).$$

For $b \in N$ we expand this by making the substitution for $\boldsymbol{v}(1)$ given by (13.1).

$$\begin{aligned}
\boldsymbol{i}_{ext}(b) &= d_b \boldsymbol{v}(b) - w_{b,1} \boldsymbol{v}(1) - \sum_{c \sim b, c \neq 1} w_{b,c} \boldsymbol{v}(c) \\
&= d_b \boldsymbol{v}(b) - w_{b,1} \frac{1}{d_1} \sum_{a \in N} w_{1,a} \boldsymbol{v}(a) - \sum_{c \sim b, c \neq 1} w_{b,c} \boldsymbol{v}(c) \\
&= d_b \boldsymbol{v}(b) - \sum_{a \in N} \frac{w_{b,1} w_{a,1}}{d_1} \boldsymbol{v}(a) - \sum_{c \sim b, c \neq 1} w_{b,c} \boldsymbol{v}(c).
\end{aligned}$$

To finish, observe that $b \in N$, so we are counting b in the middle sum above. Removing the double-count gives.

$$\boldsymbol{i}_{ext}(b) = (d_b - w_{b,1}^2/d_1)\boldsymbol{v}(b) - \sum_{a \in N, a \neq b} \frac{w_{b,1}w_{a,1}}{d_1}\boldsymbol{v}(a) - \sum_{c \sim b, c \neq 1} w_{b,c}\boldsymbol{v}(c).$$

We will show that these revised equations have two interesting properties: they are the result of applying Gaussian elimination to eliminate vertex 1, and the resulting equations are Laplacian.

Let's look at exactly how the matrix has changed. In the row for vertex b, the edge to vertex 1 was removed, and edges to every vertex $a \in N$ were added with weights $\frac{w_{b,1}w_{a,1}}{d_1}$. And, the diagonal was decreased by $\frac{w_{b,1}w_{b,1}}{d_1}$. That should look familiar to you! Overall, the star of edges based at 1 were removed, and a clique on N was added in which edge (a, b) has weight

$$\frac{w_{b,1}w_{1,a}}{d_1}.$$

If the initial graph only consisted of a star centered at 1, then the graph we produce on eliminating vertex 1 is exactly the weighted clique you considered in Homework 2. In the next section, we will see that we can use this to solve that problem.

To see that this new system of equations comes from a Laplacian, we observe that

1. It is symmetric.
2. The off-diagonal entries that have been added are negative.
3. The sum of the changes in diagonal and off-diagonal entries is zero, so the row-sum is still zero. This follows from
$$\frac{w_{b,1}^2}{d_1} - \sum_{a \in N} \frac{w_{b,1}w_{a,1}}{d_1} = 0.$$

13.4.1 In matrix form by energy

I'm now going to try doing this in terms of the quadratic form. That is, we will compute the matrix $\boldsymbol{L}_B$ so that

$$\boldsymbol{v}(B)^T \boldsymbol{L}_B \boldsymbol{v}(B) = \boldsymbol{v}^T \boldsymbol{L} \boldsymbol{v},$$

given that $\boldsymbol{v}$ is harmonic at vertex 1 and agrees with $\boldsymbol{v}(B)$ elsewhere. The quadratic form that we want to compute is thus given by

$$\begin{pmatrix} \frac{1}{d_1}\sum_{b \sim 1} w_{1,b}\boldsymbol{v}(b) \\ \boldsymbol{v}(B) \end{pmatrix}^T \boldsymbol{L} \begin{pmatrix} \frac{1}{d_1}\sum_{b \sim 1} w_{1,b}\boldsymbol{v}(b) \\ \boldsymbol{v}(B) \end{pmatrix}.$$

So that I can write this in terms of the entries of the Laplacian matrix, note that $d_1 = \boldsymbol{L}(1,1)$, and so

$$\boldsymbol{v}(1) = \frac{1}{d_1}\sum_{b \sim 1} w_{1,b}\boldsymbol{v}(b) = -(1/\boldsymbol{L}(1,1))\boldsymbol{L}(1,B)\boldsymbol{v}(B).$$

Thus, we can write the quadratic form as

$$\begin{pmatrix} -(1/\boldsymbol{L}(1,1))\boldsymbol{L}(1,B)\boldsymbol{v}(B) \\ \boldsymbol{v}(B) \end{pmatrix}^T \boldsymbol{L} \begin{pmatrix} -(1/\boldsymbol{L}(1,1))\boldsymbol{L}(1,B)\boldsymbol{v}(B) \\ \boldsymbol{v}(B) \end{pmatrix}.$$

If we expand this out, we find that it equals

$$\begin{aligned} &\boldsymbol{v}(B)^T\boldsymbol{L}(B,B)\boldsymbol{v}(B) + \boldsymbol{L}(1,1)\left(-(1/\boldsymbol{L}(1,1))\boldsymbol{L}(1,B)\boldsymbol{v}(B)\right)^2 + 2\boldsymbol{v}(1)\boldsymbol{L}(1,B)\left(-(1/\boldsymbol{L}(1,1))\boldsymbol{L}(1,B)\boldsymbol{v}(B)\right) \\ &= \boldsymbol{v}(B)^T\boldsymbol{L}(B,B)\boldsymbol{v}(B) + \left(\boldsymbol{L}(1,B)\boldsymbol{v}(B)\right)^2/\boldsymbol{L}(1,1) - 2\left(\boldsymbol{L}(1,B)\boldsymbol{v}(B)\right)^2/\boldsymbol{L}(1,1) \\ &= \boldsymbol{v}(B)^T\boldsymbol{L}(B,B)\boldsymbol{v}(B) - \left(\boldsymbol{L}(1,B)\boldsymbol{v}(B)\right)^2/\boldsymbol{L}(1,1). \end{aligned}$$

Thus,

$$\boldsymbol{L}_B = \boldsymbol{L}(B,B) - \frac{\boldsymbol{L}(B,1)\boldsymbol{L}(1,B)}{\boldsymbol{L}(1,1)}.$$

To see that this is the matrix that appears in rows and columns 2 through n when we eliminate the entries in the first column of $\boldsymbol{L}$ by adding multiples of the first row, note that we eliminate entry $\boldsymbol{L}(a,1)$ by adding $-\boldsymbol{L}(a,1)/\boldsymbol{L}(1,1)$ times the first row of the matrix to $\boldsymbol{L}(a,:)$. Doing this for all rows in $B = \{2,\ldots,n\}$ results in this formula.

We can again check that $\boldsymbol{L}_B$ is a Laplacian matrix. It is clear from the formula that it is symmetric and that the the off-diagonal entries are negative. To check that the constant vectors are in the nullspace, we can show that the quadratic form is zero on those vectors. If $\boldsymbol{v}(B)$ is a constant vector, then $\boldsymbol{v}(1)$ must equal this constant, and so $\boldsymbol{v}$ is a constant vector and the value of the quadratic form is 0.

13.5 Eliminating Many Vertices

We can of course use the same procedure to eliminate many vertices. We begin by partitioning the vertex set into *boundary* vertices B and *internal* vertices I. We can then use Gaussian elimination to eliminate all of the internal vertices. You should recall that the submatrices produced by Gaussian elimination do not depend on the order of the eliminations. So, you may conclude that the matrix $\boldsymbol{L}_B$ is uniquely defined.

Or, observe that to eliminate the entries in row $a \in B$ and columns in S, using the rows in S, we need to add those rows, $\boldsymbol{L}(S,:)$ to row $\boldsymbol{L}(a,:)$ with coefficients $\boldsymbol{c}$ so that

$$\boldsymbol{L}(a,S) + \boldsymbol{c}\boldsymbol{L}(S,S) = 0.$$

This gives

$$\boldsymbol{c} = -\boldsymbol{L}(a,S)\boldsymbol{L}(S,S)^{-1},$$

and thus row a becomes

$$\boldsymbol{L}(a,:) - \boldsymbol{L}(a,S)\boldsymbol{L}(S,S)^{-1}\boldsymbol{L}(S,:).$$

Restricting to rows and columns in B, we are left with the matrix

$$\boldsymbol{L}(B,B) - \boldsymbol{L}(B,S)\boldsymbol{L}(S,S)^{-1}\boldsymbol{L}(S,B).$$

This is called the *schur* complement on B (or with respect to S).

This is equivalent to requiring that the variables in S be harmonic. Partition a vector $\boldsymbol{v}$ into $\boldsymbol{v}(B)$ and $\boldsymbol{v}(S)$. The harmonic equations become

$$\boldsymbol{L}(S,S)\boldsymbol{v}(S) + \boldsymbol{L}(S,B)\boldsymbol{v}(B) = 0,$$

which implies

$$\boldsymbol{v}(S) = -\boldsymbol{L}(S,S)^{-1}\boldsymbol{L}(S,B)\boldsymbol{v}(B).$$

This gives

$$\boldsymbol{i}_{ext}(B) = \boldsymbol{L}(B,S)\boldsymbol{v}(S) + \boldsymbol{L}(B,B)\boldsymbol{v}(B) = -\boldsymbol{L}(B,S)\boldsymbol{L}(S,S)^{-1}\boldsymbol{L}(S,B)\boldsymbol{v}(B) + \boldsymbol{L}(B,B)\boldsymbol{v}(B),$$

and so

$$\boldsymbol{i}_{ext}(B) = \boldsymbol{L}_B\boldsymbol{v}(B), \quad \text{where } \boldsymbol{L}_B = \boldsymbol{L}(B,B) - \boldsymbol{L}(B,S)\boldsymbol{L}(S,S)^{-1}\boldsymbol{L}(S,B)\boldsymbol{v}(B).$$

13.6 Effective Resistance is a Distance

A distance is any function on pairs of vertices such that

1. $\delta(a,a) = 0$ for every vertex a,
2. $\delta(a,b) \geq 0$ for all vertices a, b,
3. $\delta(a,b) = \delta(b,a)$, and
4. $\delta(a,c) \leq \delta(a,b) + \delta(b,c)$.

We claim that the effective resistance is a distance. The only non-trivial part to prove is the triangle inequality, (4).

From the previous section, we know that it suffices to consider graphs with only three vertices: we can reduce any graph to one on just vertices a, b and c without changing the effective resistances between them.

Lemma 13.6.1. *Let a, b and c be vertices in a graph. Then*

$$\mathrm{R_{eff}}(a,b) + \mathrm{R_{eff}}(b,c) \geq \mathrm{R_{eff}}(a,c).$$

Proof. Let

$$z = w_{a,b}, y = w_{a,c}, \quad \text{and} x = w_{b,c}.$$

If we eliminate vertex c, we create an edge between vertices a and b of weight

$$\frac{xy}{x+y}.$$

Adding this to the edge that is already there produces weight $z + \frac{xy}{x+y}$, for

$$\mathrm{R_{eff}}_{a,b} = \frac{1}{z + \frac{xy}{x+y}} = \frac{1}{\frac{zx+zy+xy}{x+y}} = \frac{x+y}{zx+zy+xy}$$

Working symmetrically, we find that we need to prove that for all positive x, y, and z

$$\frac{x+y}{zx+zy+xy} + \frac{y+z}{zx+zy+xy} \geq \frac{x+z}{zx+zy+xy},$$

which is of course true.

$\square$

13.7 An interpretation of Gaussian elimination

This gives us a way of understand how Gaussian elimination solves a system of equations like $\boldsymbol{i}_{ext} = \boldsymbol{L}\boldsymbol{v}$. It constructs a sequence of graphs, $G_2, \ldots, G_n$, so that G_i is the effective network on vertices $i, \ldots, n$. It then solves for the entries of $\boldsymbol{v}$ backwards. Given $\boldsymbol{v}(i+1), \ldots, \boldsymbol{v}(n)$ and $\boldsymbol{i}_{ext}(i)$, we can solve for $\boldsymbol{v}(i)$. If $\boldsymbol{i}_{ext}(i) = 0$, then $\boldsymbol{v}(i)$ is set to the weighted average of its neighbors. If not, then we need to take $\boldsymbol{i}_{ext}(i)$ into account here and in the elimination as well. In the case in which we fix some vertices and let the rest be harmonic, there is no such complication.