Spectral Graph Theory Lecture 16

The Second Eigenvalue of Planar Graphs

Daniel A. Spielman October 24, 2018

16.1 Overview

Spectral Graph theory first came to the attention of many because of the success of using the second Laplacian eigenvector to partition planar graphs and scientific meshes [DH72, DH73, Bar82, PSL90, Sim91].

In this lecture, we will attempt to explain this success by proving, at least for planar graphs, that the second smallest Laplacian eigenvalue is small. One can then use Cheeger's inequality to prove that the corresponding eigenvector provides a good cut.

This was already known for the model case of a 2-dimensional grid. If the grid is of size $\sqrt{n}$-by-$\sqrt{n}$, then it has $\lambda_2 \approx c/n$. Cheeger's inequality then tells us that it has a cut of conductance $c/\sqrt{n}$. And, this is in fact the cut that goes right accross the middle of one of the axes, which is the cut of minimum conductance.

Theorem 16.1.1 ([ST07])**.** *Let G be a planar graph with n vertices of maximum degree d, and let λ_2 be the second-smallest eigenvalue of its Laplacian. Then,*

$$\lambda_2 \leq \frac{8d}{n}.$$

The proof will involve almost no calculation, but will use some special properties of planar graphs. However, this proof has been generalized to many planar-like graphs, including the graphs of well-shaped 3d meshes.

16.2 Geometric Embeddings

We typically upper bound λ_2 by evidencing a test vector. Here, we will upper bound λ_2 by evidencing a test embedding. The bound we apply is:

Lemma 16.2.1. *For any $d \geq 1$,*

$$\lambda_2 = \min_{\boldsymbol{v}_1,\dots,\boldsymbol{v}_n \in \mathbb{R}^d : \sum \boldsymbol{v}_i = \boldsymbol{0}} \frac{\sum_{(i,j)\in E} \left\| \boldsymbol{v}_i - \boldsymbol{v}_j \right\|^2}{\sum_i \left\| \boldsymbol{v}_i \right\|^2}. \tag{16.1}$$

Proof. Let $\boldsymbol{v}_i = (x_i, y_i, \dots, z_i)$. We note that

$$\sum_{(i,j)\in E} \left\| \boldsymbol{v}_i - \boldsymbol{v}_j \right\|^2 = \sum_{(i,j)\in E} (x_i - x_j)^2 + \sum_{(i,j)\in E} (y_i - y_j)^2 + \cdots + \sum_{(i,j)\in E} (z_i - z_j)^2.$$

Similarly,

$$\sum_i \|\boldsymbol{v}_i\|^2 = \sum_i x_i^2 + \sum_i y_i^2 + \cdots + \sum_i z_i^2.$$

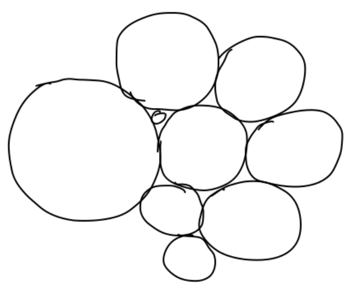
It is now trivial to show that $\lambda_2 \geq RHS$: just let $x_i = y_i = \cdots = z_i$ be given by an eigenvector of λ_2. To show that $\lambda_2 \leq RHS$, we apply my favorite inequality: $\frac{A+B+\cdots+C}{A'+B'+\cdots+C'} \geq \min\left(\frac{A}{A'}, \frac{B}{B'}, \ldots, \frac{C}{C'}\right)$, and then recall that $\sum x_i = 0$ implies

$$\frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_i x_i^2} \geq \lambda_2.$$

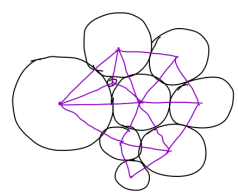
$\square$

For an example, consider the natural embedding of the square with corners $(\pm 1, \pm 1)$.

The key to applying this embedding lemma is to obtain the right embedding of a planar graph. Usually, the right embedding of a planar graph is given by Koebe's embedding theorem, which I will now explain. I begin by considering one way of generating planar graphs. Consider a set of circles $\{C_1, \ldots, C_n\}$ in the plane such that no pair of circles intersects in their interiors. Associate a vertex with each circle, and create an edge between each pair of circles that meet at a boundary. See Figure 16.2. The resulting graph is clearly planar. Koebe's embedding theorem says that *every planar graph results from such an embedding.*



(a) Circles in the plane



(b) Circles with their intersection graph

Theorem 16.2.2 (Koebe)**.** *Let $G = (V, E)$ be a planar graph. Then there exists a set of circles $\{C_1, \ldots, C_n\}$ in $\mathbb{R}^2$ that are interior-disjoint such that circle C_i touches circle C_j if and only if $(i, j) \in E$.*

This is an amazing theorem, which I won't prove today. You can find a beautiful proof in the book "Combinatorial Geometry" by Agarwal and Pach.

Such an embedding is often called a *kissing disk* embedding of the graph. From a kissing disk embedding, we obtain a natural choice of $\boldsymbol{v}_i$: the center of disk C_i. Let r_i denote the radius of this disk. We now have an easy upper bound on the numerator of (16.1): $\|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 = (r_i + r_j)^2 \leq 2r_i^2 + 2r_j^2$. On the other hand, it is trickier to obtain a lower bound on $\sum \|\boldsymbol{v}_i\|^2$. In fact, there are graphs whose kissing disk embeddings result in

$$(16.1) = \Theta(1).$$

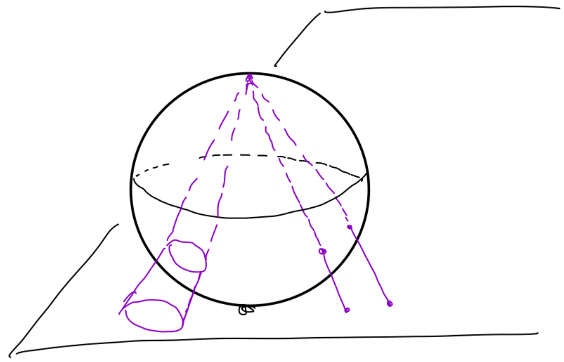
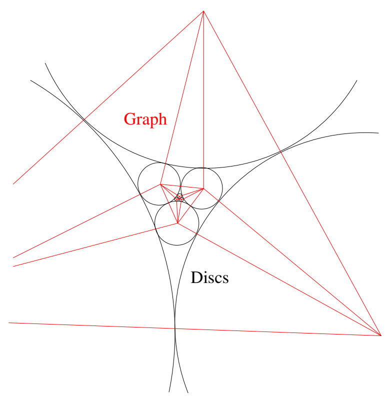


Figure 16.1: Stereographic Projection.

These graphs come from triangles inside triangles inside triangles... Such a graph is depicted below:



We will fix this problem by lifting the planar embeddings to the sphere by stereographic projection. Given a plane, $\mathbb{R}^2$, and a sphere S tangent to the plane, we can define the stereographic projection map, Π, from the plane to the sphere as follows: let $\boldsymbol{s}$ denote the point where the sphere touches the plane, and let $\boldsymbol{n}$ denote the opposite point on the sphere. For any point $\boldsymbol{x}$ on the plane, consider the line from $\boldsymbol{x}$ to $\boldsymbol{n}$. It will intersect the sphere somewhere. We let this point of intersection be $\Pi(\boldsymbol{x})$.

The fundamental fact that we will exploit about stereographic projection is that *it maps circles to circles!* So, by applying stereographic projection to a kissing disk embedding of a graph in the plane, we obtain a kissing disk embedding of that graph on the sphere. Let $D_i = \Pi(C_i)$ denote the image of circle C_i on the sphere. We will now let $\boldsymbol{v}_i$ denote the center of D_i, on the sphere.

If we had $\sum_i \boldsymbol{v}_i = \boldsymbol{0}$, the rest of the computation would be easy. For each i, $\|\boldsymbol{v}_i\| = 1$, so the denominator of (16.1) is n. Let r_i denote the straight-line distance from $\boldsymbol{v}_i$ to the boundary of D_i. We then have (see Figure 16.2)

$$\|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 \leq (r_i + r_j)^2 \leq 2r_i^2 + 2r_j^2.$$

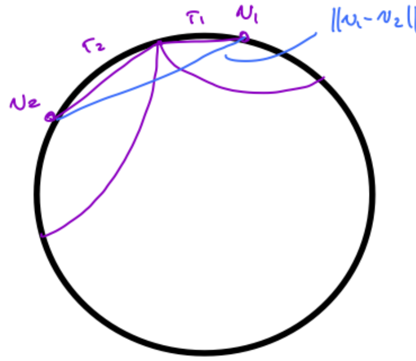


Figure 16.2: Stereographic Projection.

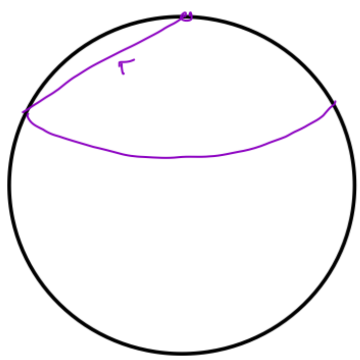


Figure 16.3: A Spherical Cap.

So, the denominator of (16.1) is at most $2d \sum_i r_i^2$. On the other hand, a theorem of Archimedes tells us that the area of the cap encircled by D_i is at exactly πr_i^2. Rather than proving it, I will convince you that it has to be true because it is true when r_i is small, it is true when the cap is a hemisphere and $r_i = \sqrt{2}$, and it is true when the cap is the whole sphere and $r_i = 2$.

As the caps are disjoint, we have

$$\sum_i \pi r_i^2 \leq 4\pi,$$

which implies that the denominator of (16.1) is at most

$$\sum_{(a,b)\in E} \left\| \boldsymbol{v}_a - \boldsymbol{v}_b \right\|^2 \leq 2r_a^2 + 2r_b^2 \leq 2d \sum_a r_a^2 \leq 8d.$$

Putting these inequalities together, we see that

$$\min_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n \in \mathbb{R}^d : \sum \boldsymbol{v}_i = \boldsymbol{0}} \frac{\sum_{(i,j)\in E} \left\| \boldsymbol{v}_i - \boldsymbol{v}_j \right\|^2}{\sum_i \left\| \boldsymbol{v}_i \right\|^2 .} \leq \frac{8d}{n}.$$

Thus, we merely need to verify that we can ensure that

$$\sum_i \boldsymbol{v}_i = \mathbf{0}. \tag{16.2}$$

Note that there is enough freedom in our construction to believe that we could prove such a thing: we can put the sphere anywhere on the plane, and we could even scale the image in the plane before placing the sphere. By carefully combining these two operations, it is clear that we can place the center of gravity of the $\boldsymbol{v}_i$s close to any point on the boundary of the sphere. It turns out that this is sufficient to prove that we can place it at the origin.

16.3 The center of gravity

We need a nice family of maps that transform our kissing disk embedding on the sphere. It is particularly convenient to parameterize these by a point ω inside the sphere. For any point α on the surface of the unit sphere, I will let Π_α denote the stereographic projection from the plane tangent to the sphere at α.

I will also define Π_α^{-1}. To handle the point $-\alpha$, I let $\Pi_\alpha^{-1}(-\alpha) = \infty$, and $\Pi_\alpha(\infty) = -\alpha$. We also define the map that dilates the plane tangent to the sphere at α by a factor a: D_α^a. We then define the following map from the sphere to itself

$$f_\omega(\boldsymbol{x}) \stackrel{\text{def}}{=} \Pi_{\omega/\|\omega\|}\left(D_{\omega/\|\omega\|}^{1-\|\omega\|}\left(\Pi_{\omega/\|\omega\|}^{-1}(\boldsymbol{x})\right)\right).$$

For $\alpha \in S$ and $\omega = a\alpha$, this map pushes everything on the sphere to a point close to α. As a approaches 1, the mass gets pushed closer and closer to α.

Instead of proving that we can achieve (16.2), I will prove a slightly simpler theorem. The proof of the theorem we really want is similar, but about just a few minutes too long for class. We will prove

Theorem 16.3.1. *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be points on the unit-sphere. Then, there exists an ω such that $\sum_i f_\omega(\boldsymbol{v}_i) = \mathbf{0}$.*

The reason that this theorem is different from the one that we want to prove is that if we apply a circle-preserving map from the sphere to itself, the center of the circle might not map to the center of the image circle.

To show that we can achieve $\sum_i \boldsymbol{v}_i = \mathbf{0}$, we will use the following topological lemma, which follows immediately from Brouwer's fixed point theorem. In the following, we let B denote the ball of points of norm less than 1, and S the sphere of points of norm 1.

Lemma 16.3.2. *If $\phi : B \to B$ be a continuous map that is the identity on S. Then, there exists an $\omega \in B$ such that*

$$\phi(\omega) = \mathbf{0}.$$

We will prove this lemma using Brouwer's fixed point theorem:

Theorem 16.3.3 (Brouwer). *If $g : B \to B$ is continuous, then there exists an $\alpha \in B$ such that $g(\alpha) = \alpha$.*

Proof of Lemma 16.3.2. Let b be the map that sends $\boldsymbol{z} \in B$ to $\boldsymbol{z}/\|\boldsymbol{z}\|$. The map b is continuous at every point other than $\mathbf{0}$. Now, assume by way of contradiction that $\mathbf{0}$ is not in the image of ϕ, and let $g(\boldsymbol{z}) = -b(\phi(\boldsymbol{z}))$. By our assumption, g is continuous and maps B to B. However, it is clear that g has no fixed point, contradicting Brouwer's fixed point theorem. $\square$

Lemma 16.3.2, was our motivation for defining the maps f_ω in terms of $\omega \in B$. Now consider setting

$$\phi(\omega) = \frac{1}{n}\sum_i f_\omega(\boldsymbol{v}_i).$$

The only thing that stops us from applying Lemma 16.3.2 at this point is that ϕ is not defined on S, because f_ω was not defined for $\omega \in S$. To fix this, we define for $\alpha \in S$

$$f_\alpha(\boldsymbol{z}) = \begin{cases} \alpha & \text{if } \boldsymbol{z} \neq -\alpha \\ -\alpha & \text{otherwise.} \end{cases}$$

We then encounter the problem that $f_\alpha(\boldsymbol{z})$ is not a continuous function of α because it is discontinuous at $\alpha = -\boldsymbol{v}_i$. But, this shouldn't be a problem because the point ω at which $\phi(\omega) = 0$ won't be on or near the boundary. The following argument makes this intuition formal.

We set

$$h_\omega(\boldsymbol{z}) = \begin{cases} 1 & \text{if } \operatorname{dist}(\omega, \boldsymbol{z}) < 2 - \epsilon, \text{ and} \\ (2 - \operatorname{dist}(\omega, \boldsymbol{z}))/\epsilon & \text{otherwise.} \end{cases}$$

Now, the function $f_\alpha(\boldsymbol{z})h_\alpha(\boldsymbol{z})$ is continuous on all of B. So, we may set

$$\phi(\omega) \stackrel{\text{def}}{=} \frac{\sum_i f_\omega(\boldsymbol{v}_i)h_\omega(\boldsymbol{v}_i)}{\sum_i h_\omega(\boldsymbol{v}_i),}$$

which is now continuous and is the identity map on S.

So, for any $\epsilon > 0$, we may now apply Lemma 16.3.2 to find an ω for which

$$\phi(\omega) = \mathbf{0}.$$

To finish the proof, we need to get rid of this ϵ. That is, we wish to show that ω is bounded away from S, say by μ, for all sufficiently small ϵ. If that is the case, then we will have $\operatorname{dist}(\omega, \boldsymbol{v}_i) \geq \mu > 0$ for all sufficiently small ϵ. So, for $\epsilon < \mu$ and sufficiently small, $h_\omega(\boldsymbol{v}_i) = 1$ for all i, and we recover the $\epsilon = 0$ case.

One can verify that this holds provided that the points $\boldsymbol{v}_i$ are distinct and there are at least 3 of them.

Finally, recall that this is not exactly the theorem we wanted to prove: this theorem deals with $\boldsymbol{v}_i$, and not the centers of caps. The difficulty with centers of caps is that they move as the caps move. However, this can be overcome by observing that the centers remain inside the caps, and move continuously with ω. For a complete proof, see [ST07, Theorem 4.2]

16.4 Further progress

This result has been improved in many ways. Jonathan Kelner [Kel06] generalized this result to graphs of bounded genus. Kelner, Lee, Price and Teng [KLPT09] obtained analogous bounds for λ_k for $k \geq 2$. Biswal, Lee and Rao [BLR10] developed an entirely new set of techniques to prove these results. Their techniques improve these bounds, and extend them to graphs that do not have K_h minors for any constant h.

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