Spectral Graph Theory Lecture 18

A simple construction of expander graphs

Daniel A. Spielman October 31, 2018

18.1 Overview

Our goal is to prove that for every $\epsilon > 0$ there is a d for which we can efficiently construct an infinite family of d-regular ϵ-expanders. I recall that these are graphs whose adjacency matrix eigenvalues satisfy $|\mu_i| \leq \epsilon d$ and whose Laplacian matrix eigenvalues satisfy $|d - \lambda_i| \leq \epsilon d$. Viewed as a function of ϵ, the d that we obtain in this construction is rather large. But, it is a constant. The challenge here is to construct infinite families with fixed d and ϵ.

Before we begin, I remind you that in Lecture 5 we showed that random generalized hybercubes were ϵ expanders of degree $f(\epsilon) \log n$, for some function f. The reason they do not solve today's problem is that their degrees depend on the number of vertices. However, today's construction will require some small expander graph, and these graphs or graphs like them can serve in that role. So that we can obtain a construction for every number of vertices n, we will exploit random generalized ring graphs. Their analysis is similar to that of random generalized hypercubes.

Claim 18.1.1. *There exists a function $f(\epsilon)$ so that for every $\epsilon > 0$ and every sufficiently large n the Cayley graph with group $\mathbb{Z}/n$ and a random set of at least $f(\epsilon) \log n$ generators is an ϵ-expander with high probability.*

I am going to present the simplest construction of expanders that I have been able to find. By "simplest", I mean optimizing the tradeoff of simplicity of construction with simplicty of analysis. It is inspired by the Zig-Zag product and replacement product constructions presented by Reingold, Vadhan and Wigderson [RVW02].

For those who want the quick description, here it is. Begin with an expander. Take its line graph. Obseve that the line graph is a union of cliques. So, replace each clique by a small expander. We need to improve the expansion slightly, so square the graph. Square one more time. Repeat.

The analysis will be simple because all of the important parts are equalities, which I find easier to understand than inequalities.

While this construction requires the choice of two expanders of constant size, it is explicit in the sense that we can obtain a simple *implict* representation of the graph: if the name of a vertex in the graph is written using b bits, then we can compute its neighbors in time polynomial in b.

18.2 Squaring Graphs

We will first show that we can obtain a family of ϵ expanders from a family of β-expanders for any $\beta < 1$. The reason is that squaring a graph makes it a better expander, although at the cost of increasing its degree.

Given a graph G, we define the graph G^2 to be the graph in which vertices u and v are connected if they are at distance 2 in G. Formally, G^2 should be a weighted graph in which the weight of an edge is the number of such paths. When first thinking about this, I suggest that you ignore the issue. When you want to think about it, I suggest treating such weighted edges as multiedges.

We may form the adjacency matrix of G^2 from the adjacency matrix of G. Let $\boldsymbol{M}$ be the adjacency matrix of G. Then $\boldsymbol{M}^2(u,v)$ is the number of paths of length 2 between u and v in G, and $\boldsymbol{M}^2(v,v)$ is always d. We will eliminate those self-loops. So,

$$\boldsymbol{M}_{G^2} = \boldsymbol{M}_G^2 - dI_n.$$

If G has no cycles of length up to 4, then all of the edges in its square will have weight 1. The following claim is immediate from this definition.

Claim 18.2.1. *The adjacency matrix eigenvalues of G^2 are precicely*

$$\mu_i^2 - d,$$

where $\mu_1, \ldots, \mu_n$ are the adjacency matrix eigenvalues of G.

Lemma 18.2.2. *If $\{G_i\}_i$ is an infinite family of d-regular β-expanders for $\beta \geq 1/\sqrt{d-1}$, then $\{G_i^2\}_i$ is an infinite family of $d(d-1)$-regular β^2 expanders.*

We remark that the case of $\beta > 1/\sqrt{d-1}$, or even larger, is the case of interest. We are not expecting to work with graphs that beat the Ramanujan bound, $2\sqrt{d-1}/d$.

Proof. For μ an adjacency matrix eigenvalue of G_i other than d, we have

$$\frac{\mu^2 - d}{d(d-1)} = \frac{\mu^2 - d}{d^2 - d} \leq \frac{\mu^2}{d^2} \leq \beta^2.$$

On the other hand, every adjacency eigenvalue of G_i^2 is at least $-d$, which is at least $-\beta^2 d(d-1)$. $\square$

So, by squaring enough times, we can convert a family of β expanders for any $\beta < 1$ into a family of ϵ expanders.

18.3 The Relative Spectral Gap

To measure the qualities of the graphs that appear in our construction, we define a quantity that we will call the *relative spectral gap* of a d-regular graph:

$$r(G) \stackrel{\text{def}}{=} \min\left(\frac{\lambda_2(G)}{d}, \frac{2d - \lambda_n}{d}\right).$$

The graphs with larger relative spectral gaps are better expanders. An ϵ-expander has relative spectral gap at least $1 - \epsilon$, and vice versa. Because we can square graphs, we know that it suffices to find an infinite family of graphs with relative spectral gap strictly greater than 0.

We now state exactly how squaring impacts the relative spectral gap of a graph.

Corollary 18.3.1. *If G has relative spectral gap β, then G^2 has relative spectral gap at least*

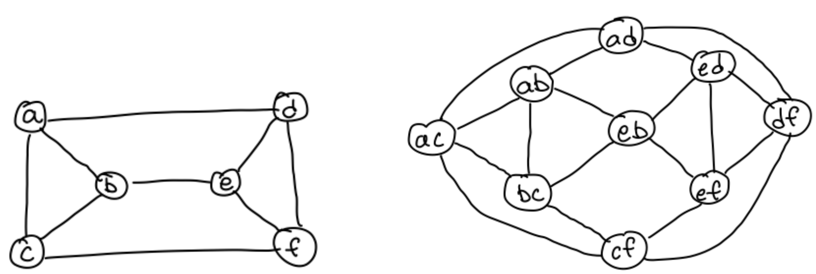
$$2\beta - \beta^2.$$

Note that when β is small, this gap is approximately 2β.

18.4 Line Graphs

Our construction will leverage small expanders to make bigger expanders. To begin, we need a way to make a graph bigger and still say something about its spectrum.

We use the *line graph* of a graph. Let $G = (V, E)$ be a graph. The line graph of G is the graph whose vertices are the edges of G in which two are connected if they share an endpoint in G. That is, $\big((u, v), (w, z)\big)$ is an edge of the line graph if one of $\{u, v\}$ is the same as one of $\{w, z\}$. The line graph is often written $L(G)$, but we won't do that in this class so that we can avoid confusion with the Laplacian.



(a) A graph (b) Its line graph.

Let G be a d-regular graph with n vertices, and let H be its line graph[1].As G has $dn/2$ edges, H has $dn/2$ vertices. Each vertex of H, say (u, v), has degree $2(d - 1)$: $d - 1$ neighbors for the other edges attached to u and $d - 1$ for v. In fact, if we just consider one vertex u in V, then all vertices in H of form (u, v) of G will be connected. That is, H contains a d-clique for every vertex in V. We see that each vertex of H is contained in exactly two of these cliques.

Here is the great fact about the spectrum of the line graph.

Lemma 18.4.1. *Let G be a d-regular graph with n vertices, and let H be its line graph. Then the spectrum of the Laplacian of H is the same as the spectum of the Laplacian of G, except that it has $dn/2 - n$ extra eigenvalues of $2d$.*

[1]If G has multiedges, which is how we interpret integer weights, then we include a vertex in the line graph for each of those multiedges. These will be connected to each other by edges of weight two—one for each vertex that they share. All of the following statements then work out.

Before we prove this lemma, we need to recall the factorization of a Laplacian as the product of the signed edge-vertex adjacency matrix times its transpose. We reserved the letter $\boldsymbol{U}$ for this matrix, and defined it by

$$\boldsymbol{U}((a,b),c) = \begin{cases} 1 & \text{if } a = c \\ -1 & \text{if } b = c \\ 0 & \text{otherwise.} \end{cases}$$

For an unweighted graph, we have

$$\boldsymbol{L}_G = \boldsymbol{U}^T \boldsymbol{U}.$$

Recall that each edge indexes one column, and that we made an arbitrary choice when we ordered the edge (a,b) rather than (b,a). But, this arbitrary choice factors out when we multiply by $\boldsymbol{U}^T$.

18.5 The Spectrum of the Line Graph

Define the matrix $|\boldsymbol{U}|$ to be the matrix obtained by replacing every entry of $\boldsymbol{U}$ by its absolute value. Now, consider $|\boldsymbol{U}|^T |\boldsymbol{U}|$. It looks just like the Laplacian, except that all of its off-diagonal entries are 1 instead of -1. So,

$$|\boldsymbol{U}|^T |\boldsymbol{U}| = \boldsymbol{D}_G + \boldsymbol{M}_G = d\boldsymbol{I} + \boldsymbol{M}_G,$$

as G is d-regular. We will also consider the matrix $|\boldsymbol{U}| |\boldsymbol{U}|^T$. This is a matrix with $nd/2$ rows and $nd/2$ columns, indexed by edges of G. The entry at the intersection of row (u,v) and column (w,z) is

$$(\boldsymbol{\delta}_u + \boldsymbol{\delta}_v)^T (\boldsymbol{\delta}_w + \boldsymbol{\delta}_z).$$

So, it is 2 if these are the same edge, 1 if they share a vertex, and 0 otherwise. That is

$$|\boldsymbol{U}| |\boldsymbol{U}|^T = 2I_{nd/2} + \boldsymbol{M}_H.$$

Moreover, $|\boldsymbol{U}| |\boldsymbol{U}|^T$ and $|\boldsymbol{U}|^T |\boldsymbol{U}|$ have the same eigenvalues, except that the later matrix has $nd/2 - n$ extra eigenvalues of 0.

Proof of Lemma 18.4.1. First, let λ_i be an eigenvalue of $\boldsymbol{L}_G$. We see that

$$\begin{aligned}
&\lambda_i \text{ is an eigenvalue of } \boldsymbol{D}_G - \boldsymbol{M}_G \implies \\
&d - \lambda_i \text{ is an eigenvalue of } \boldsymbol{M}_G \implies \\
&2d - \lambda_i \text{ is an eigenvalue of } \boldsymbol{D}_G + \boldsymbol{M}_G \implies \\
&2d - \lambda_i \text{ is an eigenvalue of } 2I_{nd/2} + \boldsymbol{M}_H \implies \\
&2(d-1) - \lambda_i \text{ is an eigenvalue of } \boldsymbol{M}_H \implies \\
&\lambda_i \text{ is an eigenvalue of } \boldsymbol{D}_H - \boldsymbol{M}_H.
\end{aligned}$$

Of course, this last matrix is the Laplacian matrix of H. We can similarly show that the extra $dn/2 - n$ zero eigenvalues of $2I_{nd/2} + \boldsymbol{M}_H$ become $2d$ in $\boldsymbol{L}_H$. □

While the line graph operation preserves λ_2, it causes the degree of the graph to grow. So, we are going to need to do more than just take line graphs to construct expanders.

Proposition 18.5.1. *Let G be a d-regular graph with $d \geq 7$ and let H be its line graph. Then,*

$$r(H) = \frac{\lambda_2(G)}{2(d-1)} \geq r(G)/2.$$

Proof. For G a d-regular graph other than K_{d+1}, $\lambda_2(G) \leq d+1$. By the Perron-Frobenius theorem (Lemma 6.A.1) $\lambda_{max}(G) \leq 2d$ (with equality if and only G is bipartite). So, $\lambda_{max}(H) = 2d$ and $\lambda_2(H) = \lambda_2(G) \leq d$. So, the term in the definition of the relative spectral gap corresponding to the largest eigenvalue of H satisfies

$$\frac{2(2d-2) - \lambda_{max}(H)}{2d-2} = \frac{2(2d-2) - 2d}{2d-2} = 1 - \frac{2}{d} \geq 5/7,$$

as $d \geq 7$. On the other hand,

$$\frac{\lambda_2(H)}{2d-2} \leq \frac{d}{2d-2} \leq 2/3.$$

As $2/3 < 5/7$,

$$\min\left(\frac{\lambda_2(H)}{2d-2}, \frac{2(2d-2) - \lambda_{max}(H)}{2d-2}\right) = \frac{\lambda_2(H)}{2d-2} = \frac{\lambda_2(G)}{2d-2} \geq r(G/2).$$

□

While the line graph of G has more vertices, its degree is higher and its relative spectral gap is approximately half that of G. We can improve the relative spectral gap by squaring. In the next section, we show how to lower the degree.

18.6 Approximations of Line Graphs

Our next step will be to construct approximations of line graphs. We already know how to approximate complete graphs: we use expanders. As line graphs are sums of complete graphs, we will approximate them by sums of expanders. That is, we replace each clique in the line graph by an expander on d vertices. Since d will be a constant in our construction, we will be able to get these small expanders from known constructions, like the random generalized ring graphs.

Let G be a d-regular graph and let Z be a graph on d vertices of degree k (we will use a low-degree expander). We define the graph

$$G \circledS{L} Z$$

to be the graph obtained by forming the edge graph of G, H, and then replacing every d-clique in H by a copy of Z. Actually, this does not uniquely define $G \text{Ⓛ} Z$, as there are many ways to replace a d-clique by a copy of Z. But, any choice will work. Note that every vertex of $G \text{Ⓛ} Z$ has degree $2k$.

Lemma 18.6.1. *Let G be a d-regular graph, let H be the line graph of G, and let Z be a k-regular α-expander. Then,*

$$(1-\alpha)\frac{k}{d}H \preccurlyeq G \circled{L} Z \preccurlyeq (1+\alpha)\frac{k}{d}H$$

Proof. As H is a sum of d-cliques, let $H_1, \ldots, H_n$ be those d-cliques. So,

$$\boldsymbol{L}_H = \sum_{i=1}^{n} \boldsymbol{L}_{H_i}.$$

Let Z_i be the graph obtained by replacing H_i with a copy of Z, on the same set of vertices. To prove the lower bound, we compute

$$\boldsymbol{L}_{G \circled{L} Z} = \sum_{i=1}^{n} \boldsymbol{L}_{Z_i} \succcurlyeq (1-\alpha)\frac{k}{d}\sum_{i=1}^{n} \boldsymbol{L}_{H_i} = (1-\alpha)\frac{k}{d}\boldsymbol{L}_H.$$

The upper bound is proved similarly. □

Corollary 18.6.2. *Under the conditions of Lemma 18.6.1,*

$$r(G \circled{L} Z) \geq \frac{1-\alpha}{2} r(G).$$

Proof. The proof is similar to the proof of Proposition 18.5.1. We have

$$\lambda_2(G \circled{L} Z) \geq (1-\alpha)\frac{k\lambda_2(G)}{d},$$

and

$$\lambda_{max}(G \circled{L} Z) \leq (1+\alpha)2k.$$

So,

$$\min\left(\lambda_2(G \circled{L} Z), 2(2k) - \lambda_{max}(G \circled{L} Z)\right) \geq \min\left((1-\alpha)\frac{k\lambda_2(G)}{d}, (1-\alpha)2k\right) = (1-\alpha)\frac{k\lambda_2(G)}{d},$$

as $\lambda_2(G) \leq d$. So,

$$r(G \circled{L} Z) \geq \frac{1}{2k}(1-\alpha)kr(G) = \frac{1-\alpha}{2}r(G).$$

□

So, the relative spectral gap of $G \circled{L} Z$ is a little less than half that of G. But, the degree of $G \circled{L} Z$ is $2k$, which we will arrange to be much less than the degree of G, d.

We will choose k and d so that squaring this graph improves its relative spectral gap, but still leaves its degree less than d. If G has relative spectral gap β, then G^2 has relative spectral gap at least

$$2\beta - \beta^2.$$

It is easy to see that when β is small, this gap is approximately 2β. This is not quite enough to compensate for the loss of $(1-\epsilon)/2$ in the corollary above, so we will have to square the graph once more.

18.7 The whole construction

To begin, we need a "small" k-regular expander graph Z on

$$d \stackrel{\text{def}}{=} (2k(2k-1))^2 - 2k(2k-1)$$

vertices. It should be an ϵ-expander for some small ϵ. I believe that $\epsilon = 1/6$ would suffice. The other graph we will need to begin our construction will be a small d-regular expander graph G_0. We use Claim 18.1.1 to establish the existence of both of these. Let β be the relative spectral gap of G_0. We will assume that β is small, but greater than 0. I believe that $\beta = 1/5$ will work. Of course, it does not hurt to start with a graph of larger relative spectral gap.

We then construct $G_0 \textcircled{L} Z$. The degree of this graph is $2k$, and its relative spectral gap is a little less than $\beta/2$. So, we square the resulting graph, to obtain

$$(G_0 \textcircled{L} Z)^2.$$

It has degree approximately $4k^2$, and relative spectral gap slightly less than β. But, for induction, we need it to be more than β. So, we square one more time, to get a relative spectral gap a little less than 2β. We now set

$$G_1 = \left((G_0 \textcircled{L} Z)^2\right)^2.$$

The graph G_1 is at least as good an approximation of a complete graph as G_0, and it has degree approximately $16k^4$. In general, we set

$$G_{i+1} = \left((G_i \textcircled{L} Z)^2\right)^2.$$

To make the inductive construction work, we need for Z to be a graph of degree k whose number of vertices equals the degree of G. This is approximately $16k^4$, and is exactly

$$(2k(2k-1))^2 - 2k(2k-1).$$

I'll now carry out the computation of relative spectral gaps with more care. Let's assume that G_0 has a relative spectral gap of $\beta \geq 4/5$, and assume, by way of induction, that $\rho(G_i) \geq 4/5$. Also assume that Z is a 1/6-expander. We then find

$$r(G_i \textcircled{L} Z) \geq (1-\epsilon)(4/5)/2 = 1/3.$$

So, $G_i \textcircled{L} Z$ is a 2/3-expander. Our analysis of graph squares then tells us that G_{i+1} is a $(2/3)^4$-expander. So,

$$r(G_{i+1}) \geq 1 - (2/3)^4 = 65/81 > 4/5.$$

By induction, we conclude that every G_i has relative spectral gap at least 4/5.

To improve their relative spectral gaps of the graphs we produce, we can just square them a few times.

18.8 Better Constructions

There is a better construction technique, called the Zig-Zag product [RVW02]. The Zig-Zag construction is a little trickier to understand, but it achieves better expansion. I chose to present the line-graph based construction because its analysis is very closely related to an analysis of the Zig-Zag product.

References

[RVW02] Omer Reingold, Salil Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. *Annals of Mathematics*, 155(1):157–187, 2002.