Spectral Graph Theory Lecture 20

The Lovàsz - Simonovits Approach to Random Walks

Daniel A. Spielman November 7, 2018

20.1 Introduction

These notes are still very rough, and will be finished later.

For a vector $\boldsymbol{f}$ and an integer k, we define $\boldsymbol{f}\{k\}$ to be the sum of the largest k entries of $\boldsymbol{f}$. For convenience, we define $\boldsymbol{f}\{0\} = 0$. Symbolically, you can define this by setting π to be a permutation for which

$$\boldsymbol{f}(\pi(1)) \geq \boldsymbol{f}(\pi(2)) \geq \ldots \geq \boldsymbol{f}(\pi(n)),$$

and then setting

$$\boldsymbol{f}\{k\} = \sum_{i=1}^{k} \boldsymbol{f}(\pi(i)).$$

For real number x between 0 and n, we define $\boldsymbol{f}\{x\}$ by making it be piece-wise linear between consecutive integers. This means that for x between integers k and $k+1$, the slope of $\boldsymbol{f}\{\}$ at x is $\boldsymbol{f}(\pi(k+1))$. As these slopes are monotone nonincreasing, the function $\boldsymbol{f}\{x\}$ is concave.

We will prove the following theorem of Lovàsz and Simonovits [LS90] on the behavior of $\boldsymbol{W}\boldsymbol{f}$.

Theorem 20.1.1. *Let $\boldsymbol{W}$ be the transition matrix of the lazy random walk on a d-regular graph with conductance at least ϕ. Let $\boldsymbol{g} = \boldsymbol{W}\boldsymbol{f}$. Then for all integers $0 \leq k \leq n$*

$$\boldsymbol{g}\{k\} \leq \frac{1}{2}\left(\boldsymbol{f}\{k - \phi h\} + \boldsymbol{f}\{k + \phi h\}\right),$$

where $h = \min(k, n-k)$.

I remark that this theorem has a very clean extension to irregular, weighted graphs. I just present this version to simplify the exposition.

We can use this theorem to bound the rate of convergence of random walks in a graph. Let $\boldsymbol{p}_t$ be the probability distribution of the walk after t steps, and plot the curves $\boldsymbol{p}_t\{x\}$. The theorem tells us that these curves lie beneath each other, and that each curve lies beneath a number of chords drawn across the previous. The walk is uniformly mixed when the curve reaches a straight line from $(0,0)$ to $(n,1)$. This theorem tells us how quickly the walks approach the straight line.

Today, we will use the theorem to prove a variant of Cheeger's inequality.

20.2 Definitions and Elementary Observations

We believe that larger conductance should imply faster mixing. In the case of Theorem 20.1.1, it should imply lower curves. This is because wider chords lie beneath narrower ones.

Claim 20.2.1. *Let $h(x)$ be a convex function, and let $z > y > 0$. Then,*

$$\frac{1}{2}\left(h(x-z)+h(x+z)\right) \leq \frac{1}{2}\left(h(x-y)+h(x+y)\right).$$

Claim 20.2.2. *Let $\boldsymbol{f}$ be a vector, let $k \in [0,n]$, and let $\alpha_1, \ldots, \alpha_n$ be numbers between 0 and 1 such that*

$$\sum_i \alpha_i = k.$$

Then,

$$\sum_i \alpha_i \boldsymbol{f}(i) \leq \boldsymbol{f}\{k\}.$$

This should be obvious, and most of you proved something like this when solving problem 2 on homework 1. It is true because the way one would maximize this sum is by setting x to 1 for the largest values.

Throughout this lecture, we will only consider lazy random walks on regular graphs. For a set S and a vertex a, we define $\gamma(a,S)$ to be the probability that a walk that is at vertex a moves to S in one step. If a is not in S, this equals one half the fraction of edges from a to S. It is one half because there is a one half probability that the walk stays at a. Similarly, if a is in S, then $\gamma(a,S)$ equals one half plus one half the fraction of edges of a that end in S.

20.3 Warm up

We warm up by proving that the curves must lie under each other.

For a vector $\boldsymbol{f}$ and a set S, we define

$$\boldsymbol{f}(S) = \sum_{a \in S} \boldsymbol{f}(a).$$

For every k there is at least one set S for which

$$\boldsymbol{f}(S) = \boldsymbol{f}\{k\}.$$

If the values of $\boldsymbol{f}$ are distinct, then the set S is unique.

Lemma 20.3.1. *Let $\boldsymbol{f}$ be a vector and let $\boldsymbol{g} = \boldsymbol{W}\boldsymbol{f}$. Then for every $x \in [0,n]$,*

$$\boldsymbol{g}\{x\} \leq \boldsymbol{f}\{x\}.$$

Proof. As the function $\boldsymbol{g}\{x\}$ is piecewise linear between integers, it suffices to prove it at integers k. Let k be an integer and let S be a set of size k for which $\boldsymbol{f}(S) = \boldsymbol{f}\{k\}$. As $\boldsymbol{g} = \boldsymbol{W}\boldsymbol{f}$,

$$\boldsymbol{g}(S) = \sum_{a \in V} \gamma(a, S)\boldsymbol{f}(a).$$

As the graph is regular,

$$\sum_{a \in V} \gamma(a, S) = k.$$

Thus, Claim 20.2.2 implies

$$\sum_{a \in V} \gamma(a, S)\boldsymbol{f}(a) \leq \boldsymbol{f}\{k\}.$$

□

20.4 The proof

Recall that the conductance of a subset of vertices S in a d-regular graph is defined to be

$$\phi(S) \stackrel{\text{def}}{=} \frac{|\partial(S)|}{d \min(|S|, n - |S|)}.$$

Our proof of the main theorem improves the previous argument by exploiting the conductance through the following lemma.

Lemma 20.4.1. *Let S be any set of k vertices. Then*

$$\sum_{a \notin S} \gamma(a, S) = (\phi(S)/2) \min(k, n - k).$$

Proof. For $a \notin S$, $\gamma(a, S)$ equals half the fraction of the edges from a that land in S. And, the number of edges leaving S equals $d\phi(S)\min(k, n - k)$. □

Lemma 20.4.2. *Let $\boldsymbol{W}$ be the transition matrix of the lazy random walk on a d-regular graph, and let $\boldsymbol{g} = \boldsymbol{W}\boldsymbol{f}$. For every set S of size k with conductance at least ϕ,*

$$\boldsymbol{g}(S) \leq \frac{1}{2}\left(\boldsymbol{f}\{k - \phi h\} + \boldsymbol{f}\{k + \phi h\}\right),$$

where $h = \min(k, n - k)$.

Proof. To ease notation, define $\gamma(a) = \gamma(a, S)$. We prove the theorem by rearranging the formula

$$\boldsymbol{g}(S) = \sum_{a \in V} \gamma(a)\boldsymbol{f}(a).$$

Recall that $\sum_{a \in V} \gamma(a) = k$.

For every vertex a define

$$\alpha(a) = \begin{cases} \gamma(a) - 1/2 & \text{if } a \in S \\ 0 & \text{if } a \notin S \end{cases} \qquad \text{and} \qquad \beta(a) = \begin{cases} 1/2 & \text{if } a \in S \\ \gamma(a) & \text{if } a \notin S. \end{cases}$$

As $\alpha(a) + \beta(a) = \gamma(a)$,

$$\boldsymbol{g}(S) = \sum_{a \in V} \alpha(a) \boldsymbol{f}(a) + \sum_{a \in V} \beta(a) \boldsymbol{f}(a).$$

We now come to the point in the argument where we exploit the laziness of the random walk, which manifests as the fact that $\gamma(a) \geq 1/2$ for $a \in S$, and so $0 \leq \alpha(a) \leq 1/2$ for all a. Similarly, $0 \leq \beta(a) \leq 1/2$ for all a. So, we can write

$$\sum_{a \in V} \alpha(a) \boldsymbol{f}(a) = \frac{1}{2} \sum_{a \in V} (2\alpha(a)) \boldsymbol{f}(a), \qquad \text{and} \qquad \sum_{a \in V} \beta(a) \boldsymbol{f}(a) = \frac{1}{2} \sum_{a \in V} (2\beta(a)) \boldsymbol{f}(a)$$

with all coefficients $2\alpha(a)$ and $2\beta(a)$ between 0 and 1. As

$$\sum_{a \in V} \beta(a) = \frac{k}{2} + \sum_{a \notin S} \gamma(a),$$

we can set

$$z = \sum_{a \notin S} \gamma(a)$$

and write

$$\sum_{a \in V} (2\alpha(a)) = k - 2z \qquad \text{and} \qquad \sum_{a \in V} (2\beta(a)) = k + 2z.$$

Lemma 20.4.1 implies that

$$z \geq \phi h / 2.$$

By Claim 20.2.2,

$$\boldsymbol{g}(S) \leq \frac{1}{2} \left(\boldsymbol{f}\{k - z\} + \boldsymbol{f}\{k + z\} \right).$$

So, Claim 20.2.1 implies

$$\boldsymbol{g}(S) \leq \frac{1}{2} \left(\boldsymbol{f}\{k - \phi h\} + \boldsymbol{f}\{k + \phi h\} \right).$$

□

Theorem 20.1.1 follows by applying Lemma 20.4.2 to sets S for which $\boldsymbol{f}(S) = \boldsymbol{f}\{k\}$, for each integer k between 0 and n.

20.5 Andersen's proof of Cheeger's inequality

Reid Andersen observed that the technique of Lovàsz and Simonovits can be used to give a new proof of Cheeger's inequality. I will state and prove the result for the special case of d-regular graphs that we consider in this lecture. But, one can of course generalize this to irregular, weighted graphs.

Theorem 20.5.1. *Let G be a d-regular graph with lazy random walk matrix $\boldsymbol{W}$, and let $\omega_2 = 1-\lambda$ be the second-largest eigenvalue of $\boldsymbol{W}$. Then there is a subset of vertices S for which*

$$\phi(S) \leq \sqrt{8\lambda}.$$

Proof. Let $\boldsymbol{\psi}$ be the eigenvector corresponding to ω_2. As $\boldsymbol{\psi}$ is orthogonal to the constant vectors, $\boldsymbol{\psi}\{n\} = 0$. Define

$$k = \arg\max_{0 \leq k \leq n} \frac{\boldsymbol{\psi}\{k\}}{\sqrt{\min(k, n-k)}}.$$

Then, set γ to be the maximum value obtained:

$$\gamma = \frac{\boldsymbol{\psi}\{k\}}{\sqrt{\min(k, n-k)}}.$$

We will assume without loss of generality that $k \leq n/2$: if it is not then we replace $\boldsymbol{\psi}$ by $-\boldsymbol{\psi}$ to make it so and obtain the same γ. Now, $\boldsymbol{\psi}\{k\} = \gamma\sqrt{k}$.

We let S be a set (there is probably only one) for which

$$\boldsymbol{\psi}(S) = \boldsymbol{\psi}\{k\}.$$

As $\boldsymbol{\psi}$ is an eigenvector with positive eigenvalue, we also know that

$$(\boldsymbol{W}\boldsymbol{\psi})(S) = \boldsymbol{W}\boldsymbol{\psi}\{k\}.$$

We also know that

$$(\boldsymbol{W}\boldsymbol{\psi})(S) = (1-\lambda)\boldsymbol{\psi}(S) = (1-\lambda)\gamma\sqrt{k}.$$

Let ϕ be the conductance of S. Lemma 20.4.2 tells us that

$$(\boldsymbol{W}\boldsymbol{\psi})(S) \leq \frac{1}{2}\left(\boldsymbol{\psi}\{k - \phi k\} + \boldsymbol{\psi}\{k + \phi k\}\right).$$

By the construction of k and γ at the start of the proof, we know this quantity is at most

$$\frac{1}{2}\left(\gamma\sqrt{k - \phi k} + \gamma\sqrt{k + \phi k}\right) = \gamma\sqrt{k}\frac{1}{2}\left(\sqrt{1-\phi} + \sqrt{1+\phi}\right).$$

Combining the inequalities derived so far yields

$$(1-\lambda) \leq \frac{1}{2}\left(\sqrt{1-\phi} + \gamma\sqrt{1+\phi}\right).$$

An examination of the Taylor series for the last terms reveals that

$$\frac{1}{2}\left(\sqrt{1-\phi} + \gamma\sqrt{1+\phi}\right) \leq 1 - \phi^2/8.$$

This implies $\lambda \geq \phi^2/8$, and thus $\phi(S) \leq \sqrt{8\lambda}$. $\square$

References

[LS90] L. Lovàsz and M. Simonovits. The mixing rate of Markov chains, an isoperimetric inequality, and computing the volume. In IEEE, editor, *Proceedings: 31st Annual Symposium on Foundations of Computer Science: October 22–24, 1990, St. Louis, Missouri*, volume 1, pages 346–354, 1109 Spring Street, Suite 300, Silver Spring, MD 20910, USA, 1990. IEEE Computer Society Press.