Spectral Graph Theory Lecture 25

Matching Polynomials of Graphs

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Disclaimer

These notes are not necessarily an accurate representation of what happened in class. The notes written before class say what I think I should say. I sometimes edit the notes after class to make them way what I wish I had said.

There may be small mistakes, so I recommend that you check any mathematically precise statement before using it in your own work.

These notes were last revised on December 7, 2015.

25.1 Overview

The coefficients of the matching polynomial of a graph count the numbers of matchings of various sizes in that graph. It was first defined by Heilmann and Lieb [HL72], who proved that it has some amazing properties, including that it is real rooted. They also proved that all root of the matching polynomial of a graph of maximum degree d are at most $2\sqrt{d-1}$. In the next lecture, we will use this fact to derive the existence of Ramanujan graphs.

Our proofs today come from a different approach to the matching polynomial that appears in the work of Godsil [God93, God81]. My hope is that someone can exploit Godsil's approach to connect the $2\sqrt{d-1}$ bound from today's lecture with that from last lecture. In today's lecture, $2\sqrt{d-1}$ appears as an upper bound on the spectral radius of a d-ary tree. Infinite d-ary trees appear as the graphs of free groups in free probability. I feel like there must be a formal relation between these that I am missing.

25.2 The Matching Polynomial

A matching in a graph $G = (V, E)$ is a subgraph of G in which every vertex has degree 1. We say that a matching has size k if it has k edges. We let

$$m_k(G)$$

denote the number of matchings in G of size k. Throughout this lecture, we let $|V| = n$. Observe that $m_1(G)$ is the number of edges in G, and that $m_{n/2}(G)$ is the number of perfect matchings in G. By convention we set $m_0(G) = 1$, as the empty set is matching with no edges. Computing the number of perfect matchings is a $\#P$-hard problem. This means that it is much harder than solving NP-hard problems, so you shouldn't expect to do it quickly on large graphs.

The *matching polynomial of* G, written $\mu_x[G]$, is

$$\mu_x[G] \stackrel{\text{def}}{=} \sum_{k=0}^{n/2} x^{n-2k}(-1)^k m_k(G).$$

Our convention that $m_0(G) = 1$ ensures that this is a polynomial of degree n.

This is a fundamental example of a polynomial that is defined so that its coefficients count something. When the "something" is interesting, the polynomial usually is as well.

25.3 Properties of the Matching Polynomial

We begin by establishing some fundamental properties of the matching polynomial. For graphs G and H on different vertex sets, we write $G \cup H$ for their disjoint union.

Lemma 25.3.1. *Let G and H be graphs on different vertex sets. Then,*

$$\mu_x[G \cup H] = \mu_x[G]\,\mu_x[H].$$

Proof. Every matching in $G \cup H$ is the union of a matchings in G and a matching in H. Thus,

$$m_k(G \cup H) = \sum_{j=0}^{k} \mu_j(G)\mu_{k-j}(H).$$

The lemma follows. □

For a a vertex of $G = (V, E)$, we write $G - a$ for the graph $G(V - \{a\})$. This notation will prove very useful when reasoning about matching polynmomials. Fix a vertex a of G, and divide the matchings in G into two classes: those that involve vertex a and those that do not. The number of matchings of size k that do not involve a is $m_k(G - a)$. On the other hand, those that do involve a connect a to one of its neighbors. To count these, we enumerate the neighbors b of a. A matching of size k that includes edge (a, b) can be written as the union of (a, b) and a matching of size $k - 1$ in $G - a - b$. So, the number of matchings that involve a is

$$\sum_{b \sim a} m_{k-1}(G - a - b).$$

So,

$$m_k(G) = m_k(G - a) + \sum_{b \sim a} m_{k-1}(G - a - b).$$

To turn this into a recurrence for $\mu_x[G]$, write

$$x^{n-2k}(-1)^k m_k(G) = x \cdot x^{n-1-2k}(-1)^k m_k(G - a) - x^{n-2-2(k-1)}(-1)^{k-1} m_{k-1}(G - a - b).$$

This establishes the following formula.

Lemma 25.3.2.

$$\mu_x\left[G\right] = x\mu_x\left[G - a\right] - \sum_{b \sim a} \mu_x\left[G - a - b\right].$$

The matching polynomials of trees are very special—they are exactly the same as the characteristic polynomial of the adjacency matrix.

Theorem 25.3.3. *Let G be a tree. Then*

$$\mu_x\left[G\right] = \chi_x(\boldsymbol{A}_G).$$

Proof. Expand

$$\chi_x(\boldsymbol{A}_G) = \det(x\boldsymbol{I} - \boldsymbol{A}_G)$$

by summing over permutations. We obtain

$$\sum_{\pi \in S_n} (-1)^{\mathsf{sgn}(\pi)} x^{|\{a : \pi(a) = a\}|} \prod_{a : \pi(a) \neq a} (-\boldsymbol{A}_G(a, \pi(a))).$$

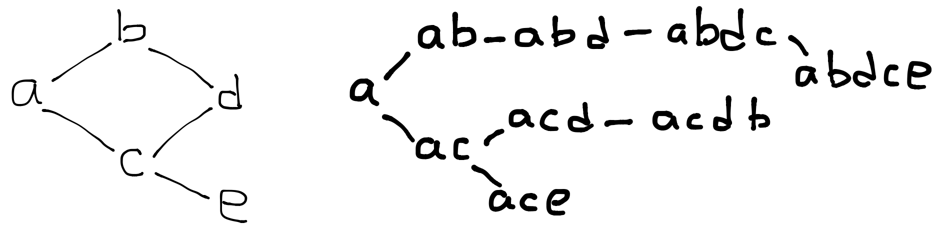
We will prove that the only permutations that contribute to this sum are those for which $\pi(\pi(a)) = a$ for every a. And, these correspond to matchings.

If π is a permutation for which there is an a so that $\pi(\pi(a)) \neq a$, then there are $a = a_1, \ldots, a_k$ with $k > 2$ so that $\pi(a_i) = a_{i+1}$ for $1 \leq i < k$, and $\pi(a_k) = a_1$. For this term to contribute, it must be the case that $\boldsymbol{A}_G(a_i, a_{i+1}) = 1$ for all i, and that $\boldsymbol{A}_G(a_k, a_1) = 1$. For $k > 2$, this would be a cycle of length k in G. However, G is a tree and so cannot have a cycle.

So, the only permutations that contribute are the *involutions*: the permutations π that are their own inverse. An involution has only fixed points and cycles of length 2. Each cycle of length 2 that contributes a nonzero term corresponds to an edge in the graph. Thus, the number of permutations with k cycles of length 2 is equal to the number of matchings with k edges. As the sign of an involution with k cycles of length 2 is $(-1)^k$, the coefficient of x^{n-2k} is $(-1)^k m_k(G)$. □

25.4 The Path Tree

Godsil proves that the matching polynomial of a graph is real rooted by proving that it divides the matching polynomial of a tree. As the matching polynomial of a tree is the same as the characteristic polynomial of its adjacency matrix, it is real rooted. Thus, the matching polynomial of the graph is as well. The tree that Godsil uses is the *path tree of* G starting at a vertex of G. For a a vertex of G, the path tree of G starting at a, written $T_a(G)$ is a tree whose vertices correspond to paths in G that start at a and do not contain any vertex twice. One path is connected to another if one extends the other by one vertex. For example, here is a graph and its path tree starting at a.



When G is a tree, $T_a(G)$ is isomorphic to G.

Godsil's proof begins by deriving a somewhat strange equality. Since I haven't yet found a better proof, I'll take this route too.

Theorem 25.4.1. *For every graph G and vertex a of G,*

$$\frac{\mu_x[G-a]}{\mu_x[G]} = \frac{\mu_x[T_a(G)-a]}{\mu_x[T_a(G)]}.$$

The term on the upper-right hand side is a little odd. It is a forrest obtained by removing the root of the tree $T_a(G)$. We may write it as a disjoint union of trees as

$$T_a(G) - a = \bigcup_{b\sim a} T_b(G-a).$$

Proof. If G is a tree, then the left and right sides are identical, and so the inequality holds. As the only graphs on less than 3 vertices are trees, the theorem holds for all graphs on at most 2 vertices. We will now prove it by induction on the number of vertices.

We may use Lemma 25.3.2 to expand the reciprocal of the left-hand side:

$$\frac{\mu_x[G]}{\mu_x[G-a]} = \frac{x\mu_x[G-a] - \sum_{b\sim a}\mu_x[G-a-b]}{\mu_x[G-a]} = x - \sum_{b\sim a}\frac{\mu_x[G-a-b]}{\mu_x[G-a]}.$$

By applying the inductive hypothesis to $G-a$, we see that this equals

$$x - \sum_{b\sim a}\frac{\mu_x[T_b(G-a)-b]}{\mu_x[T_b(G-a)]}. \tag{25.1}$$

To simplify this expression, we examine these graphs carefully. By the observtion we made before the proof,

$$T_b(G-a) - b = \bigcup_{c\sim b, c\neq a} T_c(G-a-b).$$

Similarly,

$$T_a(G) - a = \bigcup_{c\sim a} T_c(G-a),$$

which implies

$$\mu_x[T_a(G)-a] = \prod_{c\sim a}\mu_x[T_c(G-a)].$$

Let ab be the vertex in $T_a(G)$ corresponding to the path from a to b. We also have

$$\begin{aligned} T_a(G) - a - ab &= \left(\bigcup_{c \sim a, c \neq b} T_c(G - a) \right) \cup \left(\bigcup_{c \sim b, c \neq a} T_c(G - a - b) \right) \\ &= \left(\bigcup_{c \sim a, c \neq b} T_c(G - a) \right) \cup \left(T_b(G - a) - b \right). \end{aligned}$$

which implies

$$\mu_x \left[T_a(G) - a - ab \right] = \left(\prod_{c \sim a, c \neq b} \mu_x \left[T_c(G - a) \right] \right) \mu_x \left[T_b(G - a) - b \right].$$

Thus,

$$\begin{aligned} \frac{\mu_x \left[T_a(G) - a - ab \right]}{\mu_x \left[T_a(G) - a \right]} &= \frac{\left(\prod_{c \sim a, c \neq b} \mu_x \left[T_c(G - a) \right] \right) \mu_x \left[T_b(G - a) - b \right]}{\prod_{c \sim a} \mu_x \left[T_c(G - a) \right]} \\ &= \frac{\mu_x \left[T_b(G - a) - b \right]}{\mu_x \left[T_b(G - a) \right]}. \end{aligned}$$

Plugging this in to (25.1), we obtain

$$\begin{aligned} \frac{\mu_x [G]}{\mu_x [G - a]} &= x - \sum_{b \sim a} \frac{\mu_x \left[T_a(G) - a - ab \right]}{\mu_x \left[T_a(G) - a \right]} \\ &= \frac{x \mu_x \left[T_a(G) - a \right] - \sum_{b \sim a} \mu_x \left[T_a(G) - a - ab \right]}{\mu_x \left[T_a(G) - a \right]} \\ &= \frac{\mu_x \left[T_a(G) \right]}{\mu_x \left[T_a(G) - a \right]}. \end{aligned}$$

Be obtain the equality claimed in the theorem by taking the reciprocals of both sides. □

Theorem 25.4.2. *For every vertex a of G, the polynomial $\mu_x [G]$ divides the polynomial $\mu_x [T_a(G)]$.*

Proof. We again prove this by induction on the number of vertices in G, using as our base case graphs with at most 2 vertices. We then know, by induction, that for $b \sim a$,

$$\mu_x [G - a] \quad \text{divides} \quad \mu_x \left[T_b(G - a) \right].$$

As

$$T_a(G) - a = \cup_{b \sim a} T_b(G - a),$$

$$\mu_x \left[T_b(G - a) \right] \quad \text{divides} \quad \mu_x \left[T_a(G) - a \right].$$

Thus,

$$\mu_x [G - a] \quad \text{divides} \quad \mu_x \left[T_a(G) - a \right],$$

and so

$$\frac{\mu_x\left[T_a(G)-a\right]}{\mu_x\left[G-a\right]}$$

is a polynomial in x. To finish the proof, we apply Theorem 25.4.1, which implies

$$\mu_x\left[T_a(G)\right] = \mu_x\left[T_a(G)-a\right]\frac{\mu_x\left[G\right]}{\mu_x\left[G-a\right]} = \mu_x\left[G\right]\frac{\mu_x\left[T_a(G)-a\right]}{\mu_x\left[G-a\right]}.$$

□

25.5 Root bounds

If every vertex of G has degree at most d, then the same is true of $T_a(G)$. We will show that the norm of the adjacency matrix of a tree in which every vertex has degree at most d is at most $2\sqrt{d-1}$. Thus, all of the roots of the matching polynomial of a graph of maximum degree d are at most $2\sqrt{d-1}$.

Theorem 25.5.1. *Let T be a tree in which every vertex has degree at most d. Then, all eigenvalues of $\chi_x(\boldsymbol{A}_T)$ have absolute value at most $2\sqrt{d-1}$.*

Proof. Let $\boldsymbol{A}$ be the adjacency matrix of T. Choose some vertex to be the root of the tree, and define its height to be 0. For every other vertex a, define its height, $h(a)$, to be its distance to the root. Define $\boldsymbol{D}$ to be the diagonal matrix with

$$\boldsymbol{D}(a,a) = \left(\sqrt{d-1}\right)^{h(a)}.$$

Recall that the eigenvalues of $\boldsymbol{A}$ are the same as the eigenvalues of $\boldsymbol{DAD}^{-1}$. We will use the fact that all eigenvalues of a nonnegative matrix are upper bounded in absolute value by its maximum row sum.

So, we need to prove that all row sums of $\boldsymbol{DAD}^{-1}$ are at most $2\sqrt{d-1}$. There are three types of vertices to consider. First, the row of the root has up to d entries that are all $1/\sqrt{d-1}$. For $d \geq 2$, $d/\sqrt{d-1} \leq 2\sqrt{d-1}$. The intermediate vertices have one entry in their row that equals $\sqrt{d-1}$, and up to $d-1$ entries that are equal to $1/\sqrt{d-1}$, for a total of $2\sqrt{d-1}$. Finally, every leaf only has one nonzero entry in its row, and that entry equals $\sqrt{d-1}$. □

When combined with Theorem 25.4.2, this tells us that that matching polynomial of a graph with all degrees at most d has all of its roots bounded in absolute value by $2\sqrt{d-1}$.

References

[God81] C. D. Godsil. Matchings and walks in graphs. *Journal of Graph Theory*, 5(3):285–297, 1981.

[God93] Chris Godsil. *Algebraic Combinatorics*. Chapman & Hall, 1993.

[HL72] Ole J Heilmann and Elliott H Lieb. Theory of monomer-dimer systems. *Communications in Mathematical Physics*, 25(3):190–232, 1972.