Spectral Graph Theory	Lecture 25

Bipartite Ramanujan Graphs

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25.1 Overview

Margulis [Mar88] and Lubotzky, Phillips and Sarnak [LPS88] presented the first explicit constructions of infinite families of Ramanujan graphs. These had degrees $p+1$, for primes p. There have been a few other explicit constructions, [Piz90, Chi92, JL97, Mor94], all of which produce graphs of degree $q+1$ for some prime power q. Over this lecture and the next we will prove the existence of infinite families of bipartite Ramanujan of every degree. While today's proof of existence does not lend itself to an explicit construction, it is easier to understand than the presently known explicit constructions.

We think that much stronger results should be true. There is good reason to think that random d-regular graphs should be Ramanujan [MNS08]. And, Friedman [Fri08] showed that a random d-regular graph is almost Ramanujan: for sufficiently large n such a graph is a $2\sqrt{d-1}+\epsilon$ approximation of the complete graph with high probability, for every $\epsilon > 0$.

In today's lecture, we will use the method of interlacing families of polynomials to prove (half) a conjecture of Bilu and Linial [BL06] that every bipartite Ramanujan graph has a 2-lift that is also Ramanujan. This theorem comes from [MSS15a], but today's proof is informed by the techniques of [HPS15]. We will use theorems about the matching polynomials of graphs that we will prove next lecture.

In the same way that a Ramanujan graph approximates the complete graph, a bipartite Ramanujan graph approximates a complete bipartite graph. We say that a d-regular graph is a bipartite Ramanujan graph if all of its adjacency matrix eigenvalues, other than d and $-d$, have absolute value at most $2\sqrt{d-1}$. The eigenvalue of d is a consequence of being d-regular and the eigenvalue of $-d$ is a consequence of being bipartite. In particular, recall that the adjacency matrix eigenvalues of a bipartite graph are symmetric about the origin. This is a special case of the following claim, which you can prove when you have a sparse moment.

Claim 25.1.1. *The eigenvalues of a symmetric matrix of the form*

$$\begin{pmatrix} \mathbf{0} & \boldsymbol{A} \\ \boldsymbol{A}^T & \mathbf{0} \end{pmatrix}$$

are symmetric about the origin.

We remark that one can derive bipartite Ramanujan graphs from ordinary Ramanujan graphs—just take the double cover. However, we do not know any way to derive ordinary Ramanujan graphs from the bipartite ones.

As opposed to reasoning directly about eigenvalues, we will work with characteristic polynomials. For a matrix $\boldsymbol{M}$, we write its characteristic polynomial in the variable x as

$$\chi_x(\boldsymbol{M}) \stackrel{\text{def}}{=} \det(x\boldsymbol{I} - \boldsymbol{M}).$$

25.2 2-Lifts

We saw 2-lifts of graphs in Problem 3 from Problem Set 2:

> We define a *signed adjacency matrix* of G to be a symmetric matrix $\boldsymbol{S}$ with the same nonzero pattern as the adjacency matrix $\boldsymbol{A}$, but such that each nonzero entry is either 1 or -1.
>
> We will use it to define a graph $G^{\boldsymbol{S}}$. Like the double-cover, the graph $G^{\boldsymbol{S}}$ will have two vertices for every vertex of G and two edges for every edge of G. For each edge $(u, v) \in E$, if $\boldsymbol{S}(u, v) = -1$ then $G^{\boldsymbol{S}}$ has the two edges
>
> $$(u_1, v_2) \quad \text{and} \quad (v_1, u_2),$$
>
> just like the double-cover. If $\boldsymbol{S}(u, v) = 1$, then $G^{\boldsymbol{S}}$ has the two edges
>
> $$(u_1, v_1) \quad \text{and} \quad (v_2, u_2).$$
>
> You should check that $G^{-\boldsymbol{A}}$ is the double-cover of G and that $G^{\boldsymbol{A}}$ consists of two disjoint copies of G.
>
> Prove that the eigenvalues of the adjacency matrix of $G^{\boldsymbol{S}}$ are the union of the eigenvalues of $\boldsymbol{A}$ and the eigenvalues of $\boldsymbol{S}$.

The graphs $G^{\boldsymbol{S}}$ that we form this way are called 2-lifts of G.

Bilu and Linial [BL06] conjectured that every d-regular graph G has a signed adjacency matrix $\boldsymbol{S}$ so that $\|\boldsymbol{S}\| \leq 2\sqrt{d-1}$. This would give a simple procedure for constructing infinite families of Ramanujan graphs. We would begin with any small d-regular Ramanujan graph, such as the complete graph on $d + 1$ vertices. Then, given any d-regular Ramanujan graph we could construct a new Ramanujan graph on twice as many vertices by using $G^{\boldsymbol{S}}$ where $\|\boldsymbol{S}\| \leq 2\sqrt{d-1}$.

We will prove something close to their conjecture.

Theorem 25.2.1. *Every d-regular graph G has a signed adjacency matrix $\boldsymbol{S}$ for which the minimum eigenvalue of $\boldsymbol{S}$ is at least $-2\sqrt{d-1}$.*

We can use this theorem to build infinite families of bipartite Ramanujan graphs, because their eigenvalues are symmetric about the origin. Thus, if $\mu_n \geq -2\sqrt{d-1}$, then we know that $|\mu_i| \leq 2\sqrt{d-1}$ for all $1 < i < n$. Note that every 2-lift of a bipartite graph is also a bipartite graph.

25.3 Random 2-Lifts

We will prove Theorem 25.2.1 by considering a random 2-lift. In particular, we consider the expected characteristic polynomial of a random signed adjacency matrix $\boldsymbol{S}$:

$$\mathbb{E}_{\boldsymbol{S}}\left[\chi_x(\boldsymbol{S})\right]. \tag{25.1}$$

Godsil and Gutman [GG81] proved that this is equal to the matching polynomial of G! We will learn more about the matching polynomial next lecture.

For now, we just need the following bound on its zeros which was proved by Heilmann and Lieb [HL72].

Theorem 25.3.1. *The eigenvalues of the matching polynomial of a graph of maximum degree at most d are real and have absolute value at most $2\sqrt{d-1}$.*

Now that we know that the smallest zero of (25.1) is at least $-2\sqrt{d-1}$, all we need to do is to show that there is some signed adjacency matrix whose smallest eigenvalue is at least this bound. This is not necessarily as easy as it sounds, because the smallest zero of the average of two polynomials is not necessarily related to the smallest zeros of those polynomials. We will show that, in this case, it is.

25.4 Laplacianized Polynomials

Instead of directly reasoning about the characteristic polynomials of signed adjacency matrices $\boldsymbol{S}$, we will work with characteristic polynomials of $d\boldsymbol{I} - \boldsymbol{S}$. It suffices for us to prove that there exists an $\boldsymbol{S}$ for which the largest eigenvalue of $d\boldsymbol{I} - \boldsymbol{S}$ is at most $d + 2\sqrt{d-1}$.

Fix an ordering on the m edges of the graph, associate each $\boldsymbol{S}$ with a vector $\sigma \in \{\pm 1\}^m$, and define

$$p_\sigma(x) = \chi_x(d\boldsymbol{I} - \boldsymbol{S}).$$

The expected polynomial is the average of all these polynomials.

We define two vectors for each edge in the graph. If the ith edge is (a,b), then we define

$$\boldsymbol{v}_{i,\sigma_i} = \boldsymbol{\delta}_a - \sigma_i \boldsymbol{\delta}_b.$$

For every $\sigma \in \{\pm 1\}^m$, we have

$$\sum_{i=1}^m \boldsymbol{v}_{i,\sigma_i} \boldsymbol{v}_{i,\sigma_i}^T = d\boldsymbol{I} - \boldsymbol{S},$$

where $\boldsymbol{S}$ is the signed adjacency matrix corresponding to σ. So, for every $\sigma \in \{\pm 1\}^m$,

$$p_\sigma(x) = \chi_x\left(\sum_{i=1}^m \boldsymbol{v}_{i,\sigma_i} \boldsymbol{v}_{i,\sigma_i}^T\right).$$

25.5 Interlacing Families of Polynomials

Here is the problem we face. We have a large family of polynomials, say $p_1(x), \dots, p_m(x)$, for which we know each p_i is real-rooted and that their sum is real rooted. We would like to show that there is some polynomial p_i whose largest zero is at most the largest zero of the sum. This is not true in general. But, it is true in our case because the polynomials form an *interlacing family*.

For a polynomial $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$ of degree n and a polynomial $q(x) = \prod_{i=1}^{n-1}(x - \mu_i)$ of degree $n-1$, we say that $q(x)$ *interlaces* $p(x)$ if

$$\lambda_n \leq \mu_{n-1} \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \mu_1 \leq \lambda_1.$$

If $r(x) = \prod_{i=1}^{n}(x - \mu_i)$ has degree n, we write $r(x) \to p(x)$ if

$$\mu_n \leq \lambda_n \leq \mu_{n-1} \leq \cdots \leq \lambda_2 \leq \mu_1 \leq \lambda_1.$$

That is, if the zeros of p and r interlace, with the zeros of p being larger. We also make these statements if they hold of positive multiples of p, r and q.

The following lemma gives the examples of interlacing polynomials that motivate us.

Lemma 25.5.1. *Let $\boldsymbol{A}$ be a symmetric matrix and let $\boldsymbol{v}$ be a vector. For a real number t let*

$$p_t(x) = \chi_x(\boldsymbol{A} + t\boldsymbol{v}\boldsymbol{v}^T).$$

Then, for $t > 0$, $p_0(x) \to p_t(x)$ and there is a monic[1] degree $n-1$ polynomial $q(x)$ so that for all t

$$p_t(x) = \chi_x(\boldsymbol{A}) - tq(x).$$

Proof. The fact that $p_0(x) \to p_t(x)$ for $t > 0$ follows from the Courant-Fischer Theorem.

We first establish the existence of $q(x)$ in the case that $\boldsymbol{v} = \boldsymbol{\delta}_1$. As the matrix $t\boldsymbol{\delta}_1\boldsymbol{\delta}_1^T$ is zeros everywhere except for the element t in the upper left entry and the determinant is linear in each entry of the matrix,

$$\chi_x(\boldsymbol{A} + t\boldsymbol{\delta}_1\boldsymbol{\delta}_1^T) = \det(x\boldsymbol{I} - \boldsymbol{A} - t\boldsymbol{\delta}_1\boldsymbol{\delta}_1^T) = \det(x\boldsymbol{I} - \boldsymbol{A}) - t\det(x\boldsymbol{I}^{(1)} - \boldsymbol{A}^{(1)}) = \chi_x(\boldsymbol{A}) - t\chi_x(\boldsymbol{A}^{(1)}),$$

where $\boldsymbol{A}^{(1)}$ is the submatrix of $\boldsymbol{A}$ obtained by removing its first row and column. The polynomial $q(x) = \chi_x(\boldsymbol{A}^{(1)})$ has degree $n-1$.

For arbitrary, $\boldsymbol{v}$, let $\boldsymbol{Q}$ be a rotation matrix for which $\boldsymbol{Q}\boldsymbol{v} = \boldsymbol{\delta}_1$. As determinants, and thus characteristic polynomials, are unchanged by multiplication by rotation matrices,

$$\begin{aligned}\chi_x(\boldsymbol{A} + t\boldsymbol{v}\boldsymbol{v}^T) &= \chi_x(\boldsymbol{Q}(\boldsymbol{A} + t\boldsymbol{v}\boldsymbol{v}^T)\boldsymbol{Q}^T) \\ &= \chi_x(\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T + t\boldsymbol{\delta}_1\boldsymbol{\delta}_1^T)) = \chi_x(\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T) - tq(x) = \chi_x(\boldsymbol{A}) - tq(x),\end{aligned}$$

for some $q(x)$ of degree $n-1$. □

[1] A *monic* polynomial is one whose leading coefficient is 1.

For a polynomial p, let $\lambda_{max}(p)$ denote its largest zero. When polynomials interlace, we can relate the largest zero of their sum to the largest zero of at least one of them.

Lemma 25.5.2. *Let $p_1(x)$, $p_2(x)$ and $r(x)$ be polynomials so that $r(x) \to p_i(x)$. Then, $r(x) \to p_1(x) + p_2(x)$ and there is an $i \in \{1, 2\}$ for which*

$$\lambda_{max}(p_i) \leq \lambda_{max}(p_1 + p_2).$$

Proof. Let μ_1 be the largest zero of $r(x)$. As each polynomial $p_i(x)$ has a positive leading coefficient, each is eventually positive and so is their sum. As each has exactly one zero that is at least μ_1 each is nonpositive at μ_1, and the same is also true of their sum. Let λ be the largest zero of $p_1 + p_2$. We have established that $\lambda \geq \mu_1$.

If $p_i(\lambda) = 0$ for some i, then we are done. If not, there is an i for which $p_i(\lambda) > 0$. As p_i only has one zero larger than μ_1, and it is eventually positive, the largest zero of p_i must be less than λ. $\square$

If $p_1, \ldots, p_m$ are polynomials such that there exists an $r(x)$ for which $r(x) \to p_i(x)$ for all i, then these polynomials are said to have a *common interlacing*. Such polynomials satisfy the natural generalization of Lemma 25.5.2.

The polynomials $p_\sigma(x)$ do not all have a common interlacing. However, they satisfy a property that is just as useful: they form an *interlacing family*. Rather than defining these in general, we will just explain the special case we need for today's theorem.

We define polynomials that correspond to fixing the signs of the first k edges and then choosing the rest at random. We indicate these by shorter sequences $\sigma \in \{\pm 1\}^k$. For $k < m$ and $\sigma \in \{\pm 1\}^k$ we define

$$p_\sigma(x) \stackrel{\text{def}}{=} \mathbb{E}_{\rho \in \{\pm 1\}^{n-k}} \left[p_{\sigma,\rho}(x) \right].$$

So,

$$p_\emptyset(x) = \mathbb{E}_{\sigma \in \{\pm 1\}^m} \left[p_\sigma(x) \right].$$

We view the strings σ, and thus the polynomials p_σ, as vertices in a complete binary tree. The nodes with σ of length m are the leaves, and $\emptyset$ corresponds to the root. For σ of length less than n, the children of σ are $(\sigma, 1)$ and $(\sigma, -1)$. We call such a pair of nodes *siblings*. We will eventually prove in Lemma 25.6.1 that all the polynomials $p_\sigma(x)$ are real rooted and in Corollary 25.6.2 that every pair of siblings has a common interlacing.

But first, we show that this implies that there is a leaf indexed by $\sigma \in \{\pm 1\}^m$ for which

$$\lambda_{max}(p_\sigma) \leq \lambda_{max}(p_\emptyset).$$

This implies Theorem 25.2.1, as we know from Theorem 25.3.1 that $\lambda_{max}(p_\emptyset) \leq d + 2\sqrt{d-1}$.

Lemma 25.5.3. *There is a $\sigma \in \{\pm 1\}^m$ for which*

$$\lambda_{max}(p_\sigma) \leq \lambda_{max}(p_\emptyset).$$

Proof. Corollary 25.6.2 and Lemma 25.5.2 imply that every non-leaf node in the tree has a child whose largest zero is at most the largest zero of that node. Starting at the root of the tree, we find a node whose largest zero is at most the largest zero of $p_\emptyset$. We then proceed down the tree until we reach a leaf, at each step finding a node labeled by a polynomial whose largest zero is at most the largest zero of the previous polynomial. The leaf we reach, σ, satisfies the desired inequality. $\square$

25.6 Common Interlacings

We can now use Lemmas 25.5.1 and 25.5.2 to show that every $\sigma \in \{\pm 1\}^{m-1}$ has a child (σ, s) for which $\lambda_{max}(p_{\sigma,s}) \leq \lambda_{max}(p_\sigma)$. Let

$$\boldsymbol{A} = \sum_{i=1}^{m-1} \boldsymbol{v}_{i,\sigma_i} \boldsymbol{v}_{i,\sigma_i}^T.$$

The children of σ, $(\sigma, 1)$ and $(\sigma, -1)$ have polynomials $p_{(\sigma,1)}$ and $p_{(\sigma,-1)}$ that equal

$$\chi_x(\boldsymbol{A} + \boldsymbol{v}_{m,1}\boldsymbol{v}_{m,1}^T) \quad \text{and} \quad \chi_x(\boldsymbol{A} + \boldsymbol{v}_{m,-1}\boldsymbol{v}_{m,-1}^T).$$

By Lemma 25.5.1, $\chi_x(\boldsymbol{A}) \to \chi_x(\boldsymbol{A} + \boldsymbol{v}_{m,s}\boldsymbol{v}_{m,s}^T)$ for $s \in \{\pm 1\}$, and Lemma 25.5.2 implies that there is an s for which the largest zero of $p_{(\sigma,s)}$ is at most the largest zero of their average, which is p_σ.

To extend this argument to nodes higher up in the tree, we will prove the following statement.

Lemma 25.6.1. *Let $\boldsymbol{A}$ be a symmetric matrix and let $\boldsymbol{w}_{i,s}$ be vectors for $1 \leq i \leq k$ and $s \in \{0, 1\}$. Then the polynomial*

$$\sum_{\rho \in \{0,1\}^k} \chi_x\left(\boldsymbol{A} + \sum_{i=1}^{k} \boldsymbol{w}_{i,\rho_i} \boldsymbol{w}_{i,\rho_i}^T\right)$$

is real rooted, and for each $s \in \{0, 1\}$,

$$\sum_{\rho \in \{0,1\}^k} \chi_x\left(\boldsymbol{A} + \sum_{i=1}^{k-1} \boldsymbol{w}_{i,\rho_i} \boldsymbol{w}_{i,\rho_i}^T\right) \to \sum_{\rho \in \{0,1\}^k} \chi_x\left(\boldsymbol{A} + \sum_{i=1}^{k-1} \boldsymbol{w}_{i,\rho_i} \boldsymbol{w}_{i,\rho_i}^T + \boldsymbol{w}_{k,s}\boldsymbol{w}_{k,s}^T\right).$$

Corollary 25.6.2. *For every $k < n$ and $\sigma \in \{\pm 1\}^k$, the polynomials $p_{\sigma,s}(x)$ for $s \in \{\pm 1\}$ are real rooted and have a common interlacing.*

25.7 Real Rootedness

To prove Lemma 25.6.1, we use the following two lemmas which are known collectively as Obreschkoff's Theorem [Obr63].

Lemma 25.7.1. *Let p and q be polynomials of degree n and $n - 1$, and let $p_t(x) = p(x) - tq(x)$. If p_t is real rooted for all $t \in \mathbb{R}$, then q interlaces p.*

Proof Sketch. Recall that the roots of a polynomial are continuous functions of its coefficients, and thus the roots of p_t are continuous functions of t. We will use this fact to obtain a contradiction.

For simplicity,[2] I just consider the case in which all of the roots of p and q are distinct. If they are not, one can prove this by dividing out their common divisors.

If p and q do not interlace, then p must have two roots that do not have a root of q between them. Let these roots of p be λ_{i+1} and λ_i. Assume, without loss of generality, that both p and q are positive between these roots. We now consider the behavior of p_t for positive t.

As we have assumed that the roots of p and q are distinct, q is positive at these roots, and so p_t is negative at λ_{i+1} and λ_i. If t is very small, then p_t will be close to p in value, and so there must be some small t_0 for which $p_{t_0}(x) > 0$ for some $\lambda_{i+1} < x < \lambda_i$. This means that p_{t_0} must have two roots between λ_{i+1} and λ_i.

As q is positive on the entire closed interval $[\lambda_{i+1}, \lambda_i]$, when t is large p_t will be negative on this entire interval, and thus have no roots inside. As we vary t between t_0 and infinity, the two roots at t_0 must vary continuously and cannot cross λ_{i+1} or λ_i. This means that they must become complex, contradicting our assumption that p_t is always real rooted. $\square$

Lemma 25.7.2. *Let p and q be polynomials of degree n and $n-1$ that interlace and have positive leading coefficients. For every $t > 0$, define $p_t(x) = p(x) - tq(x)$. Then, $p_t(x)$ is real rooted and*

$$p(x) \rightarrow p_t(x).$$

Proof Sketch. For simplicity, I consider the case in which all of the roots of p and q are distinct. One can prove the general case by dividing out the common repeated roots.

To see that the largest root of p_t is larger than λ_1, note that $q(x)$ is positive for all $x > \mu_1$, and $\lambda_1 > \mu_1$. So, $p_t(\lambda_1) = p(\lambda_1) - tq(\lambda_1) < 0$. As p_t is monic, it is eventually positive and it must have a root larger than λ_1.

We will now show that for every $i \geq 1$, p_t has a root between λ_{i+1} and λ_i. As this gives us $d-1$ more roots, it accounts for all d roots of p_t. For i odd, we know that $q(\lambda_i) > 0$ and $q(\lambda_{i+1}) < 0$. As p is zero at both of these points, $p_t(\lambda_i) > 0$ and $p_t(\lambda_{i+1}) < 0$, which means that p_t has a root between λ_i and λ_{i+1}. The case of even i is similar. $\square$

Lemma 25.7.3. *Let $p_0(x)$ and $p_1(x)$ be degree n monic polynomials for which there is a third polynomial $r(x)$ Such that*

$$r(x) \rightarrow p_0(x) \quad \textit{and} \quad r(x) \rightarrow p_1(x).$$

Then

$$r(x) \rightarrow (1/2)p_0(x) + (1/2)p_1(x),$$

and the latter is a real rooted polynomial.

Sketch. Assume for simplicity that all the roots of r are distinct and different from the roots of p_0 and p_1. Let $\mu_n < \mu_{n-1} < \cdots < \mu_1$ be the roots of r. Our assumptions imply that both p_0 and p_1

[2] I thank Sushant Sachdeva for helping me work out this particularly simple proof.

are negative at μ_i for odd i and positive for even i. So, the same is true of their average. This tells us that their average must have at least $n-1$ real roots between μ_n and μ_1. As their average is monic, it must be eventually positive and so must have a root larger than μ_1. That accounts for all n of its roots. $\square$

Proof of Lemma 25.6.1. We prove this by induction on k. Assuming that we have proved it for $k-1$, we now prove it for k. Let $\boldsymbol{u}$ be any vector and let $t \in \mathbb{R}$. Define

$$p_t(x) = \sum_{\rho \in \{0,1\}^k} \chi_x \left(\boldsymbol{A} + \sum_{i=1}^{k-1} \boldsymbol{w}_{i,\rho_i} \boldsymbol{w}_{i,\rho_i}^T + t\boldsymbol{u}\boldsymbol{u}^T \right).$$

By Lemma 25.5.1, we can express this polynomial in the form

$$p_t(x) = p_0(x) - tq(x),$$

where q has positive leading coefficient and degree $n-1$. By absorbing $t\boldsymbol{u}\boldsymbol{u}^T$ into $\boldsymbol{A}$ we may use induction on k to show that $p_t(x)$ is real rooted for all t. Thus, Lemma 25.7.1 implies that $q(x)$ interlaces $p_0(x)$, and Lemma 25.7.2 tells us that for $t > 0$

$$p_0(x) \to p_t(x).$$

So, we may conclude that for every $s \in \{\pm 1\}$,

$$\sum_{\rho \in \{0,1\}^{k-1}} \chi_x \left(\boldsymbol{A} + \sum_{i=1}^{k-1} \boldsymbol{w}_{i,\rho_i} \boldsymbol{w}_{i,\rho_i}^T \right) \to \sum_{\rho \in \{0,1\}^k} \chi_x \left(\boldsymbol{A} + \sum_{i=1}^{k-1} \boldsymbol{w}_{i,\rho_i} \boldsymbol{w}_{i,\rho_i}^T + \boldsymbol{w}_{k,s} \boldsymbol{w}_{k,s}^T \right).$$

So, Lemma 25.7.3 implies that

$$\sum_{\rho \in \{0,1\}^{k-1}} \chi_x \left(\boldsymbol{A} + \sum_{i=1}^{k-1} \boldsymbol{w}_{i,\rho_i} \boldsymbol{w}_{i,\rho_i}^T \right) \to \sum_{\rho \in \{0,1\}^k} \chi_x \left(\boldsymbol{A} + \sum_{i=1}^{k} \boldsymbol{w}_{i,\rho_i} \boldsymbol{w}_{i,\rho_i}^T \right)$$

and that the latter polynomial is real rooted. $\square$

25.8 Conclusion

The major open problem left by this work is establishing the existence of regular (non-bipartite) Ramanujan graphs. The reason we can not prove this using the techniques in this lecture is that the interlacing techniques only allow us to reason about the largest or smallest eigenvalue of a matrix, but not both.

To see related papers establishing the existence of Ramanujan graphs, see [MSS15b, HPS15]. For a survey on this and related material, see [MSS14].

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