Spectral Graph Theory	Lecture 26
Bipartite Ramanujan Graphs of Every Degree	
Daniel A. Spielman	December 9, 2015

Disclaimer

These notes are not necessarily an accurate representation of what happened in class. The notes written before class say what I think I should say. I sometimes edit the notes after class to make them way what I wish I had said.

There may be small mistakes, so I recommend that you check any mathematically precise statement before using it in your own work.

These notes were last revised on December 9, 2015.

These notes are still very rough.

26.1 Overview

In today's lecture, we will prove the existence of infinite families of bipartite Ramanujan of every degree. We do this by proving (half) a conjecture of Bilu and Linial [BL06] that every bipartite Ramanujan graph has a 2-lift that is also Ramanujan.

Today's theorem comes from [MSS15], and the proof is informed by the techniques of [HPS15]. We will use theorems about the matching polynomials of graphs that we proved last lecture.

26.2 2-Lifts

We saw 2-lifts of graphs in Problem 4 from Problem Set 2:

> We define a *signed adjacency matrix* of G to be a symmetric matrix $\boldsymbol{S}$ with the same nonzero pattern as the adjacency matrix $\boldsymbol{A}$, but such that each nonzero entry is either 1 or -1.
>
> We will use it to define a graph $G^{\boldsymbol{S}}$. Like the double-cover, the graph $G^{\boldsymbol{S}}$ will have two vertices for every vertex of G and two edges for every edge of G. For each edge $(u, v) \in E$, if $\boldsymbol{S}(u, v) = -1$ then $G^{\boldsymbol{S}}$ has the two edges
>
> $$(u_1, v_2) \quad \text{and} \quad (v_1, u_2),$$

just like the double-cover. If $\boldsymbol{S}(u,v) = 1$, then $G^{\boldsymbol{S}}$ has the two edges

$$(u_1, v_1) \quad \text{and} \quad (v_2, u_2).$$

You should check that $G^{-\boldsymbol{A}}$ is the double-cover of G and that $G^{\boldsymbol{A}}$ consists of two disjoint copies of G.

Prove that the eigenvalues of the adjacency matrix of $G^{\boldsymbol{S}}$ are the union of the eigenvalues of $\boldsymbol{A}$ and the eigenvalues of $\boldsymbol{S}$.

The graphs $G^{\boldsymbol{S}}$ that we form this way are called 2-lifts of G.

For your convenience, I now recall the solution to this problem.

Let $\boldsymbol{A}_+$ be the matrix with entries

$$\boldsymbol{A}_+(u,v) = \begin{cases} 1 & \text{if } \boldsymbol{S}(u,v) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\boldsymbol{A}_- = -(\boldsymbol{S} - \boldsymbol{A}_+)$. So,

$$\boldsymbol{S} = \boldsymbol{A}_+ - \boldsymbol{A}_- \quad \text{and} \quad \boldsymbol{A} = \boldsymbol{A}_+ + \boldsymbol{A}_-.$$

The adjacency matrix of $G^{\boldsymbol{S}}$ can be expressed in terms of these matrices as

$$\boldsymbol{A}^{\boldsymbol{S}} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{A}_+ & \boldsymbol{A}_- \\ \boldsymbol{A}_- & \boldsymbol{A}_+ \end{pmatrix}.$$

Let $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_n$ be an orthonormal basis of eigenvectors of $\boldsymbol{A}$ of eigenvalues $\lambda_1, \dots, \lambda_n$, and let $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_n$ be an orthonormal basis of eigenvectors of $\boldsymbol{S}$ of eigenvalues $\mu_1, \dots, \mu_n$. We will prove that the vectors

$$\boldsymbol{\psi}_i^+ \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{\psi}_i \\ \boldsymbol{\psi}_i \end{pmatrix} \quad \text{and} \quad \boldsymbol{\phi}_i^- \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{\phi}_i \\ -\boldsymbol{\phi}_i \end{pmatrix}$$

are an orthogonal basis of $2n$ eigenvectors of $\boldsymbol{A}^{\boldsymbol{S}}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_n$.

For $i \neq j$, it is immediately clear that $\boldsymbol{\psi}_i^+$ and $\boldsymbol{\psi}_j^+$ are orthogonal, and that $\boldsymbol{\phi}_i^+$ and $\boldsymbol{\phi}_j^+$ are orthogonal. Also, for every i and j,

$$(\boldsymbol{\psi}_i^+)^T \boldsymbol{\phi}_j^- = \begin{pmatrix} \boldsymbol{\psi}_i \\ \boldsymbol{\psi}_i \end{pmatrix}^T \begin{pmatrix} \boldsymbol{\phi}_i \\ -\boldsymbol{\phi}_i \end{pmatrix} = \boldsymbol{\psi}_i^T \boldsymbol{\phi}_j - \boldsymbol{\psi}_i^T \boldsymbol{\phi}_j = 0.$$

To show that these are the eigenvectors with the claimed eigenvalues, compute

$$\boldsymbol{A}^{\boldsymbol{S}} \boldsymbol{\psi}_i^+ = \begin{pmatrix} \boldsymbol{A}_+ & \boldsymbol{A}_- \\ \boldsymbol{A}_- & \boldsymbol{A}_+ \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_i \\ \boldsymbol{\psi}_i \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_+ \boldsymbol{\psi}_i + \boldsymbol{A}_- \boldsymbol{\psi}_i \\ \boldsymbol{A}_- \boldsymbol{\psi}_i + \boldsymbol{A}_+ \boldsymbol{\psi}_i \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} \boldsymbol{\psi}_i \\ \boldsymbol{A} \boldsymbol{\psi}_i \end{pmatrix} = \lambda_i \begin{pmatrix} \boldsymbol{\psi}_i \\ \boldsymbol{\psi}_i \end{pmatrix},$$

and

$$\boldsymbol{A}^{\boldsymbol{S}} \boldsymbol{\phi}_i^- = \begin{pmatrix} \boldsymbol{A}_+ & \boldsymbol{A}_- \\ \boldsymbol{A}_- & \boldsymbol{A}_+ \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi}_i \\ -\boldsymbol{\phi}_i \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_+ \boldsymbol{\phi}_i - \boldsymbol{A}_- \boldsymbol{\phi}_i \\ -\boldsymbol{A}_- \boldsymbol{\phi}_i + \boldsymbol{A}_+ \boldsymbol{\phi}_i \end{pmatrix} = \begin{pmatrix} \boldsymbol{S} \boldsymbol{\phi}_i \\ \boldsymbol{S} \boldsymbol{\phi}_i \end{pmatrix} = \mu_i \begin{pmatrix} \boldsymbol{\phi}_i \\ \boldsymbol{\phi}_i \end{pmatrix}.$$

Bilu and Linial [**?**] conjectured that every d-regular graph G has a signed adjacency matrix $\boldsymbol{S}$ so that $\|\boldsymbol{S}\| \leq 2\sqrt{d-1}$. This would give a simple procedure for constructing infinite families of Ramanujan graphs. We would begin with any small d-regular Ramanujan graph, such as the complete graph on $d+1$ vertices. Then, given any d-regular Ramanujan graph we could construct a new Ramanujan graph on twice as many vertices by using $G^{\boldsymbol{S}}$ where $\|\boldsymbol{S}\| \leq 2\sqrt{d-1}$.

We will prove something close to their conjecture.

Theorem 26.2.1. *Every d-regular graph G has a signed adjacency matrix $\boldsymbol{S}$ for which the maximum eigenvalue of $\boldsymbol{S}$ is at most $2\sqrt{d-1}$.*

We can use this theorem to build infinite families of bipartite Ramanujan graphs, because their eigenvalues are symmetric about the origin. Thus, if $\mu_2 \leq 2\sqrt{d-1}$, then we know that $|\mu_i| \leq 2\sqrt{d-1}$ for all $1 < i < n$. Note that the 2-lift of a bipartite graph is also a bipartite graph.

26.3 Random 2-Lifts

We will prove Theorem 26.2.1 by considering a random 2-lift, and then applying the method of interlacing polynomials. In particular, we consider

$$\mathbb{E}\chi_x(\boldsymbol{S}). \tag{26.1}$$

Godsil and Gutman [GG81] proved that this is equal to the matching polynomial of G!

Lemma 26.3.1. *Let G be a graph and let $\boldsymbol{S}$ be a uniform random signed adjacency matrix of G. Then,*

$$\mathbb{E}\chi_x(\boldsymbol{S}) = \mu_x[G].$$

Proof. Expand the expected characterstic polynomial as

$$\begin{aligned}
\mathbb{E}\chi_x(\boldsymbol{S}) &= \mathbb{E}\det(x\boldsymbol{I} - \boldsymbol{S}) \\
&= \mathbb{E}\sum_{\pi \in S_n} (-1)^{\mathsf{sgn}(\pi)} x^{|\{a:\pi(a)=a\}|} \prod_{a:\pi(a)\neq a} (\boldsymbol{S}(a,\pi(a))). \\
&= \sum_{\pi \in S_n} (-1)^{\mathsf{sgn}(\pi)} x^{|\{a:\pi(a)=a\}|} \mathbb{E} \prod_{a:\pi(a)\neq a} (\boldsymbol{S}(a,\pi(a))).
\end{aligned}$$

As $\mathbb{E}\boldsymbol{S}(a,\pi(a)) = 0$ for every a so that $\pi(a) \neq a$, the only way we can get a nonzero contribution from a permutation π is if for all a so that $\pi(a) \neq a$,

a. $(a, \pi(a)) \in E$, and

b. $\pi(\pi(a)) = a$.

The latter condition guarantees that whenever $\boldsymbol{S}(a, \pi(a))$ appears in the product, $\boldsymbol{S}(\pi(a), a)$ does as well. As these entries are constrained to be the same, their product is 1.

Thus, the only permtuations that count are the involuations. As we saw last lecture, these correspond exactly to the matchings in the graph. □

Thus, we know that the largest root of (26.1) is at most $2\sqrt{d-1}$. So, all we need to do is to show that there is some signed adjacency matrix whose largest eigenvalue is at most this bound. We do this via the method of interlacing polynomials.

To this end, choose an ordering on the m edges of the graph. We can now associate each $\boldsymbol{S}$ with a vector $\sigma \in \{\pm 1\}^m$. Define

$$p_\sigma = \chi_x(\boldsymbol{S}).$$

The expected polynomial is the average of all these polynomials.

To form an interlacing family, we will form a tree that has the polynomials p_σ at the leaves. The intermediate nodes will correspond to choices of the first couple signs. That is, for $k < m$ and $\sigma \in \{\pm 1\}^k$ we define

$$p_\sigma(x) \stackrel{\text{def}}{=} \mathbb{E}_{\rho \in \{\pm 1\}^{n-k}} p_{\sigma,\rho}(x).$$

So, $p_\emptyset$ is the polynomial at the root of the tree. It remains to show that all pairs of siblings in the tree have a common interlacing.

Polynomials indexed by σ and τ are siblings if σ and τ have the same length, and only differ in their last index. To show that they have a common interlacing, we recall a few results from Lecture 22.

Lemma 26.3.2. *[Lemma 22.3.3] Let $\boldsymbol{A}$ be an n-dimensional symmetric matrix and let $\boldsymbol{v}$ be a vector. Let*

$$p_t(x) = \chi_x(\boldsymbol{A} + t\boldsymbol{v}\boldsymbol{v}^T).$$

Then there is a degree $n-1$ polynomial $q(x)$ so that

$$p_t(x) = \chi_x(\boldsymbol{A}) - tq(x).$$

Lemma 26.3.3. *[Lemma 22.3.2] Let p and q be polynomials of degree n and $n-1$, and let $p_t(x) = p(x) - tq(x)$. If p_t is real rooted for all $t \in \mathbb{R}$, then p and q interlace.*

Lemma 26.3.4. *[Lemma 22.3.1] Let p and q be polynomials of degree n and $n-1$ that interlace and have positive leading coefficients. For every $t > 0$, define $p_t(x) = p(x) - tq(x)$. Then, $p_t(x)$ is real rooted and*

$$p(x) \to p_t(x).$$

Lemma 26.3.5. *Let $p_0(x)$ and $p_1(x)$ be two degree n monic polynomials for which there is a third polynomial $r(x)$ that has the same degree as p_0 and p_1 and so that*

$$p_0(x) \to r(x) \quad \text{and} \quad p_1(x) \to r(x).$$

Then for all $0 \le s \le 1$,

$$p_s(x) \stackrel{\text{def}}{=} sp_1(x) + (1-s)p_0(x)$$

is a real rooted polynomial.

Theorem 26.3.6. *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ be independently distributed random n-dimensional vectors and let $\boldsymbol{A}$ be a symmetric n-dimensional matrix. Then, the polynomial*

$$\mathbb{E}\chi_x\left(\boldsymbol{A} + \sum_{i=1}^{k} \boldsymbol{v}_i \boldsymbol{v}_i^T\right)$$

is real rooted. Moreover, for every vector $\boldsymbol{u}$ in the support of $\boldsymbol{v}_k$, all the polynomials

$$\mathbb{E}\chi_x\left(\boldsymbol{A} + \boldsymbol{u}\boldsymbol{u}^T + \sum_{i=1}^{k-1} \boldsymbol{v}_i \boldsymbol{v}_i^T\right)$$

have a common interlacing.

Proof. We prove this by induction on k. Assuming that we have proved it for k, we now prove it for $k+1$. Let $\boldsymbol{u}$ be any vector and let $t \in \mathbb{R}$. Define

$$p_t(x) = \mathbb{E}\chi_x\left(\boldsymbol{A} + t\boldsymbol{u}\boldsymbol{u}^T + \sum_{i=1}^{k} \boldsymbol{v}_i \boldsymbol{v}_i^T\right).$$

By Lemma 26.3.2, we can express this polynomial in the form

$$p_t(x) = p_0(x) - tq(x),$$

where q has degree $n-1$. By induction, we know that $p_t(x)$ is real rooted for all t. Thus, Lemma 26.3.3 implies that $q(x)$ interlaces $p_0(x)$, and Lemma 26.3.4 tells us that for $t > 0$

$$p_0(x) \to p_t(x).$$

So, we may conclude that for every vector $\boldsymbol{u}$,

$$\mathbb{E}\chi_x\left(\boldsymbol{A} + \sum_{i=1}^{k} \boldsymbol{v}_i \boldsymbol{v}_i^T\right) \to \mathbb{E}\chi_x\left(\boldsymbol{A} + \boldsymbol{u}\boldsymbol{u}^T + \sum_{i=1}^{k} \boldsymbol{v}_i \boldsymbol{v}_i^T\right).$$

We now apply this result with each $\boldsymbol{u}$ from the support of $\boldsymbol{v}_{k+1}$ to conclude (via Lemma) that

$$\mathbb{E}\chi_x\left(\boldsymbol{A} + \sum_{i=1}^{k} \boldsymbol{v}_i \boldsymbol{v}_i^T\right) \to \mathbb{E}\chi_x\left(\boldsymbol{A} + \boldsymbol{v}_{k+1}\boldsymbol{v}_{k+1}^T + \sum_{i=1}^{k} \boldsymbol{v}_i \boldsymbol{v}_i^T\right),$$

and that the latter polynomial is real rooted. □

To apply this theorem to the matrices $\boldsymbol{S}$, we must write them as a sum of outer products of random vectors. While we cannot do this, we can do something just as good. For each edge (a, b) of G, let $\boldsymbol{v}_{a,b}$ be the random vector that is $\boldsymbol{\delta}_a - \boldsymbol{\delta}_b$ with probability $1/2$ and $\boldsymbol{\delta}_a + \boldsymbol{\delta}_b$ with probability $1/2$. The random matrix $\boldsymbol{S}$ is distributed according to

$$\sum_{(a,b)\in E} \boldsymbol{v}_{a,b}\boldsymbol{v}_{a,b}^T - d\boldsymbol{I}.$$

Subtracting $d\boldsymbol{I}$ shifts the roots by d, and so does not impact any results we have proved about interlacing or real rootedness.

References

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