Spectral Graph Theory	Lecture 26

Matching Polynomials of Graphs

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26.1 Overview

The coefficients of the matching polynomial of a graph count the numbers of matchings of various sizes in that graph. It was first defined by Heilmann and Lieb [HL72], who proved that it has some amazing properties, including that it is real rooted. They also proved that all roots of the matching polynomial of a graph of maximum degree d are at most $2\sqrt{d-1}$. Our proofs today come from a different approach to the matching polynomial that appears in the work of Godsil [God93, God81]. A theorem of Godsil and Gutman [GG81] implies that the expected characteristic polynomial of a randomly signed adjacency matrix is the matching polynomial of a graph. Last lecture we used these results to establish the existence of infinite families of bipartite Ramanujan graphs.

26.2 $2\sqrt{d-1}$

We begin by explaining where the number $2\sqrt{d-1}$ comes from: it is an upper bound on the eigenvalues of a tree of maximum degree at most d. One can also show that the largest eigenvalue of an d-ary tree approaches $2\sqrt{d-1}$ as the depth of the tree (and number of vertices) increases.

We prove this statement in two steps. The first is similar to proofs we saw at the beginning of the semester.

Lemma 26.2.1. *Let $\boldsymbol{M}$ be a (not necessarily symmetric) nonnegative matrix. Let $s = \|M\mathbf{1}\|_\infty$ be the maximum row sum of $\boldsymbol{M}$. Then, $|\lambda| \leq s$ for every eigenvalue of $\boldsymbol{M}$.*

Proof. Let $\boldsymbol{M}\boldsymbol{\psi} = \lambda\boldsymbol{\psi}$, and let a be an entry of $\boldsymbol{\psi}$ of largest absolute value. Then,

$$
\begin{aligned}
|\lambda|\,|\boldsymbol{\psi}(a)| &= |\lambda\boldsymbol{\psi}(a)| \\
&= |(\boldsymbol{M}\boldsymbol{\psi})(a)| \\
&= \left|\sum_b \boldsymbol{M}(b,a)\boldsymbol{\psi}(a)\right| \\
&\leq \left|\sum_b \boldsymbol{M}(b,a)\right| |\boldsymbol{\psi}(a)| \\
&\leq s\,|\boldsymbol{\psi}(a)|\,.
\end{aligned}
$$

This implies $|\lambda| \leq s$. $\square$

Theorem 26.2.2. *Let T be a tree in which every vertex has degree at most d. Then, all eigenvalues of $\chi_x(\boldsymbol{M}_T)$ have absolute value at most $2\sqrt{d-1}$.*

Proof. Let $\boldsymbol{M}$ be the adjacency matrix of T. Choose some vertex to be the root of the tree, and define its height to be 0. For every other vertex a, define its height, $h(a)$, to be its distance to the root. Define $\boldsymbol{D}$ to be the diagonal matrix with

$$\boldsymbol{D}(a,a) = \left(\sqrt{d-1}\right)^{h(a)}.$$

Recall that the eigenvalues of $\boldsymbol{M}$ are the same as the eigenvalues of $\boldsymbol{D}\boldsymbol{M}\boldsymbol{D}^{-1}$. We will use the fact that all eigenvalues of a nonnegative matrix are upper bounded in absolute value by its maximum row sum.

So, we need to prove that all row sums of $\boldsymbol{D}\boldsymbol{M}\boldsymbol{D}^{-1}$ are at most $2\sqrt{d-1}$. There are three types of vertices to consider. First, the row of the root has up to d entries that are all $1/\sqrt{d-1}$. For $d \geq 2$, $d/\sqrt{d-1} \leq 2\sqrt{d-1}$. The intermediate vertices have one entry in their row that equals $\sqrt{d-1}$, and up to $d-1$ entries that are equal to $1/\sqrt{d-1}$, for a total of $2\sqrt{d-1}$. Finally, every leaf only has one nonzero entry in its row, and that entry equals $\sqrt{d-1}$. $\square$

26.3 The Matching Polynomial

A matching in a graph $G=(V,E)$ is a subgraph of G in which every vertex has degree 1. We say that a matching has size k if it has k edges. We let

$$m_k(G)$$

denote the number of matchings in G of size k. Throughout this lecture, we let $|V| = n$. Observe that $m_1(G)$ is the number of edges in G, and that $m_{n/2}(G)$ is the number of perfect matchings in G. By convention we set $m_0(G) = 1$, as the empty set is matching with no edges. Computing the number of perfect matchings is a $\#P$-hard problem [Val79]. This means that it is much harder than solving NP-hard problems, so you shouldn't expect to do it quickly on large graphs.

The *matching polynomial of* G, written $\mu_x[G]$, is

$$\mu_x[G] \stackrel{\text{def}}{=} \sum_{k=0}^{n/2} x^{n-2k}(-1)^k m_k(G).$$

Our convention that $m_0(G) = 1$ ensures that this is a polynomial of degree n.

This is a fundamental example of a polynomial that is defined so that its coefficients count something. When the "something" is interesting, the polynomial usually is as well.

Godsil and Gutman [GG81] proved that this is equal to the matching polynomial of G!

Lemma 26.3.1. *Let G be a graph and let $\boldsymbol{S}$ be a uniform random signed adjacency matrix of G. Then,*

$$\mathbb{E}\left[\chi_x(\boldsymbol{S})\right] = \mu_x[G].$$

Proof. Expand the expected characterstic polynomial as

$$
\begin{aligned}
\mathbb{E}\left[\chi_x(\boldsymbol{S})\right] &= \mathbb{E}\left[\det(x\boldsymbol{I} - \boldsymbol{S})\right] \\
&= \mathbb{E}\left[\det(x\boldsymbol{I} + \boldsymbol{S})\right] \\
&= \mathbb{E}\left[\sum_{\pi \in S_n} \mathsf{sgn}(\pi) x^{|\{a : \pi(a) = a\}|} \prod_{a : \pi(a) \neq a} (\boldsymbol{S}(a, \pi(a))).\right] \\
&= \sum_{\pi \in S_n} \mathsf{sgn}(\pi) x^{|\{a : \pi(a) = a\}|} \mathbb{E}\left[\prod_{a : \pi(a) \neq a} (\boldsymbol{S}(a, \pi(a)))\right].
\end{aligned}
$$

As $\mathbb{E}\left[\boldsymbol{S}(a, \pi(a))\right] = 0$ for every a so that $\pi(a) \neq a$, the only way we can get a nonzero contribution from a permutation π is if for all a so that $\pi(a) \neq a$,

a. $(a, \pi(a)) \in E$, and

b. $\pi(\pi(a)) = a$.

The latter condition guarantees that whenever $\boldsymbol{S}(a, \pi(a))$ appears in the product, $\boldsymbol{S}(\pi(a), a)$ does as well. As these entries are constrained to be the same, their product is 1.

Thus, the only permtuations that count are the involuations (the permutations in which all cycles have length 1 or 2). These correspond exactly to the matchings in the graph. Finally, the sign of an involution is exactly its number of two-cycles, which is exactly its number of edges. □

We will prove that the matching polynomial of every d-regular graph divides the matching polynomial of a larger tree of maximum degree d.

The matching polynomials of trees are very special—they are exactly the same as the characteristic polynomial of the adjacency matrix.

Theorem 26.3.2. *Let G be a tree and let $\boldsymbol{M}$ be its adjacency matrix. Then*

$$\mu_x\left[G\right] = \chi_x(\boldsymbol{M}).$$

Proof. Expand

$$\chi_x(\boldsymbol{M}) = \det(x\boldsymbol{I} - \boldsymbol{M})$$

by summing over permutations. We obtain

$$\sum_{\pi \in S_n} \mathsf{sgn}(\pi) x^{|\{a : \pi(a) = a\}|} \prod_{a : \pi(a) \neq a} (-\boldsymbol{M}(a, \pi(a))).$$

We will prove that the only permutations that contribute to this sum are those for which $\pi(\pi(a)) = a$ for every a. And, these correspond to matchings.

If π is a permutation for which there is an a so that $\pi(\pi(a)) \neq a$, then there are $a = a_1, \dots, a_k$ with $k > 2$ so that $\pi(a_i) = a_{i+1}$ for $1 \leq i < k$, and $\pi(a_k) = a_1$. For this term to contribute, it must be the case that $\boldsymbol{M}(a_i, a_{i+1}) = 1$ for all i, and that $\boldsymbol{M}(a_k, a_1) = 1$. For $k > 2$, this would be a cycle of length k in G. However, G is a tree and so cannot have a cycle.

So, the only permutations that contribute are the *involutions*: the permutations π that are their own inverse. An involution has only fixed points and cycles of length 2. Each cycle of length 2 that contributes a nonzero term corresponds to an edge in the graph. Thus, the number of permutations with k cycles of length 2 is equal to the number of matchings with k edges. As the sign of an involution with k cycles of length 2 is $(-1)^k$, the coefficient of x^{n-2k} is $(-1)^k m_k(G)$. □

26.4 Properties of the Matching Polynomial

We begin by establishing some fundamental properties of the matching polynomial. For graphs G and H on different vertex sets, we write $G \cup H$ for their disjoint union.

Lemma 26.4.1. *Let G and H be graphs on different vertex sets. Then,*

$$\mu_x [G \cup H] = \mu_x [G] \, \mu_x [H] \, .$$

Proof. Every matching in $G \cup H$ is the union of a matching in G and a matching in H. Thus,

$$m_k(G \cup H) = \sum_{j=0}^{k} m_j(G) m_{k-j}(H).$$

The lemma follows. □

For a a vertex of $G = (V, E)$, we write $G - a$ for the graph $G(V - \{a\})$. This notation will prove very useful when reasoning about matching polynmomials. Fix a vertex a of G, and divide the matchings in G into two classes: those that involve vertex a and those that do not. The number of matchings of size k that do not involve a is $m_k(G - a)$. On the other hand, those that do involve a connect a to one of its neighbors. To count these, we enumerate the neighbors b of a. A matching of size k that includes edge (a, b) can be written as the union of (a, b) and a matching of size $k - 1$ in $G - a - b$. So, the number of matchings that involve a is

$$\sum_{b \sim a} m_{k-1}(G - a - b).$$

This gives a recurrence for the number of matchings of size k in G:

$$m_k(G) = m_k(G - a) + \sum_{b \sim a} m_{k-1}(G - a - b).$$

To turn this into a recurrence for $\mu_x [G]$, write

$$x^{n-2k}(-1)^k m_k(G) = x \cdot x^{(n-1)-2k}(-1)^k m_k(G - a) - x^{(n-2)-2(k-1)}(-1)^{k-1} m_{k-1}(G - a - b).$$

This establishes the following formula.

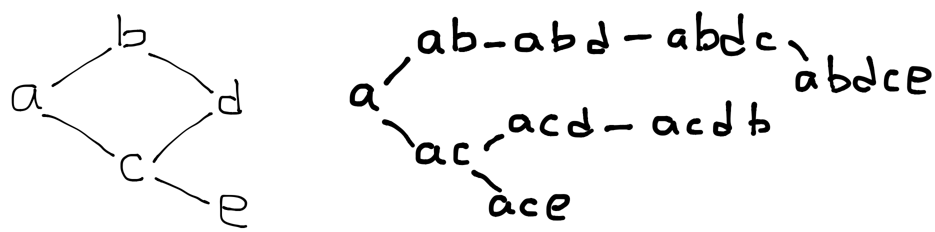
Lemma 26.4.2.

$$\mu_x [G] = x\mu_x [G - a] - \sum_{b \sim a} \mu_x [G - a - b] .$$

26.5 The Path Tree

Godsil proves that the matching polynomial of a graph is real rooted by proving that it divides the matching polynomial of a tree. Moreover, the maximum degree of vertices in the tree is at most the maximum degree of vertices in the graph. As the matching polynomial of a tree is the same as its characteristic polynomial, and all zeros of the characteristic polynomial of a tree of maximum degree at most d have absolute value at most $2\sqrt{d-1}$, all the zeros of the matching polynomial of a d-regular graph have absolute value at most $2\sqrt{d-1}$.

The tree that Godsil uses is the *path tree of* G starting at a vertex of G. For a a vertex of G, the path tree of G starting at a, written $T_a(G)$ is a tree whose vertices correspond to paths in G that start at a and do not contain any vertex twice. One path is connected to another if one extends the other by one vertex. For example, here is a graph and its path tree starting at a.



When G is a tree, $T_a(G)$ is isomorphic to G.

Godsil's proof begins by deriving a somewhat strange equality. Since I haven't yet found a better proof, I'll take this route too.

Theorem 26.5.1. *For every graph G and vertex a of G,*

$$\frac{\mu_x [G]}{\mu_x [G - a]} = \frac{\mu_x [T_a(G)]}{\mu_x [T_a(G) - a]}.$$

The term on the upper-right hand side is a little odd. It is a forrest obtained by removing the root of the tree $T_a(G)$. We may write it as a disjoint union of trees as

$$T_a(G) - a = \bigcup_{b \sim a} T_b(G - a).$$

Before proving this, we use it to prove our main theorem.

Theorem 26.5.2. *For every vertex a of G, the polynomial $\mu_x [G]$ divides the polynomial $\mu_x [T_a(G)]$.*

Proof. We prove this by induction on the number of vertices in G, using as our base case graphs with at most 2 vertices. We then know, by induction, that for $b \sim a$,

$$\mu_x[G-a] \quad \text{divides} \quad \mu_x[T_b(G-a)].$$

As

$$T_a(G) - a = \cup_{b\sim a} T_b(G-a),$$

$$\mu_x[T_b(G-a)] \quad \text{divides} \quad \mu_x[T_a(G)-a].$$

Thus,

$$\mu_x[G-a] \quad \text{divides} \quad \mu_x[T_a(G)-a],$$

and so

$$\frac{\mu_x[T_a(G)-a]}{\mu_x[G-a]}$$

is a polynomial in x. To finish the proof, we apply Theorem 26.5.1, which implies

$$\mu_x[T_a(G)] = \mu_x[T_a(G)-a]\frac{\mu_x[G]}{\mu_x[G-a]} = \mu_x[G]\frac{\mu_x[T_a(G)-a]}{\mu_x[G-a]}.$$

□

Proof of Theorem 26.5.1. If G is a tree, then the left and right sides are identical, and so the equality holds. As the only graphs on less than 3 vertices are trees, the theorem holds for all graphs on at most 2 vertices. We will now prove it by induction on the number of vertices.

We may use Lemma 26.4.2 to expand the the left-hand side:

$$\frac{\mu_x[G]}{\mu_x[G-a]} = \frac{x\mu_x[G-a] - \sum_{b\sim a}\mu_x[G-a-b]}{\mu_x[G-a]} = x - \sum_{b\sim a}\frac{\mu_x[G-a-b]}{\mu_x[G-a]}.$$

By applying the inductive hypothesis to $G-a$, we see that this equals

$$x - \sum_{b\sim a}\frac{\mu_x[T_b(G-a)-b]}{\mu_x[T_b(G-a)]}. \tag{26.1}$$

To simplify this expression, we examine these graphs carefully. By the observtion we made before the proof,

$$T_b(G-a) - b = \bigcup_{c\sim b, c\neq a} T_c(G-a-b).$$

Similarly,

$$T_a(G) - a = \bigcup_{c\sim a} T_c(G-a),$$

which implies

$$\mu_x[T_a(G)-a] = \prod_{c\sim a}\mu_x[T_c(G-a)].$$

Let ab be the vertex in $T_a(G)$ corresponding to the path from a to b. We also have

$$\begin{aligned} T_a(G) - a - ab &= \left(\bigcup_{c \sim a, c \neq b} T_c(G-a) \right) \cup \left(\bigcup_{c \sim b, c \neq a} T_c(G-a-b) \right) \\ &= \left(\bigcup_{c \sim a, c \neq b} T_c(G-a) \right) \cup (T_b(G-a)-b) \,. \end{aligned}$$

which implies

$$\mu_x\left[T_a(G)-a-ab\right] = \left(\prod_{c \sim a, c \neq b} \mu_x\left[T_c(G-a)\right] \right) \mu_x\left[T_b(G-a)-b\right].$$

Thus,

$$\begin{aligned} \frac{\mu_x\left[T_a(G)-a-ab\right]}{\mu_x\left[T_a(G)-a\right]} &= \frac{\left(\prod_{c \sim a, c\neq b} \mu_x\left[T_c(G-a)\right]\right) \mu_x\left[T_b(G-a)-b\right]}{\prod_{c \sim a} \mu_x\left[T_c(G-a)\right]} \\ &= \frac{\mu_x\left[T_b(G-a)-b\right]}{\mu_x\left[T_b(G-a)\right]}. \end{aligned}$$

Plugging this in to (26.1), we obtain

$$\begin{aligned} \frac{\mu_x[G]}{\mu_x[G-a]} &= x - \sum_{b \sim a} \frac{\mu_x\left[T_a(G)-a-ab\right]}{\mu_x\left[T_a(G)-a\right]} \\ &= \frac{x\mu_x\left[T_a(G)-a\right] - \sum_{b \sim a} \mu_x\left[T_a(G)-a-ab\right]}{\mu_x\left[T_a(G)-a\right]} \\ &= \frac{\mu_x\left[T_a(G)\right]}{\mu_x\left[T_a(G)-a\right]}. \end{aligned}$$

□

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