Graphs and Networks **out:** September 9, 2009

Problem Set 1

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1 Homework Policy

You are allowed to discuss the problems in groups of two or three, but you must write up the solutions on your own. If you do work with anyone, you should acknowledge your collaborators.

2 Adjacency Matrices

1. Let $G = (V, E)$ be a d-regular, bipartite graph. Prove that $-d$ is an eigenvalue of A_G.
2. Let $G = (V, E)$ be a d-regular graph. Prove that if $-d$ is an eigenvalue of A_G, then G is bipartite.

3 Normalized Adjacency Matrices

The Normalized Adjacency Matrix, M_G, of a graph $G = (V, E, w)$, is defined by

$$M_G = D_G^{-1/2} A_G D_G^{-1/2}.$$

We will denote its eigenvalues by

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n.$$

3. Let $G = (V, E)$ be a connected, unweighted graph. Prove that $\mu_1 = 1$. Describe the corresponding eigenvector as a function of the degrees of the vertices.
4. Prove that $\mu_n = -1$ if and only if the graph G is bipartite.

4 Star Graphs

5. Compute the eigenvalues and eigenvectors of the adjacency matrix of the star graph S_n.
6. Do the same for the normalized adjacency matrix.

5 Matrix Scaling

7. Prove that Theorem 3.3.2 follows from Lemma 3.3.3. I recommend considering the matrix $\sum_{i=0}^{n-1} A^i$. If you like, you may assume that A is symmetric. Alternatively, one can prove this by taking a limit.

8. For a matrix A, let $\min(A)$ denote the minimum entry of A. Define the matrix function

$$\gamma(A) = \frac{\min(A)}{\max(A\mathbf{1})}.$$

   Let $\boldsymbol{s} = A\mathbf{1}$. In the notation used in the proof of Lemma 3.3.3, prove that

$$\phi(\boldsymbol{s}) \leq (1 - \gamma(A))\phi(\mathbf{1}).$$

9. In the proof of Lemma 3.3.3, we studied the transformation of a matrix given by

$$f(A) = (\mathtt{diag}(A\mathbf{1}))^{-1}\ A\ (\mathtt{diag}(A\mathbf{1}))\,.$$

   Let $f^{(k)}(A)$ denote the result of applying the operation f k times. Prove that for every positive matrix A there exists an $\epsilon > 0$ such that

$$\gamma\left(f^{(k)}(A)\right) > \epsilon,$$

   for all $k > 0$.

Taken together, these statements show that if A has no zero entries, then $f^{(k)}(A)$ approaches a matrix with uniform row-sums.

6 Research Problem

10. Find a significantly faster algorithm for computing a $\boldsymbol{v}$ such that $\phi(\boldsymbol{v}) < \epsilon$.