Spectral Graph Theory		**out:** September 25, 2009
	Problem Set 2	
Lecturer: Daniel A. Spielman		**due:** October 14, 2009

1 Homework Policy

You are allowed to discuss the problems in groups of two or three, but you must write up the solutions on your own. If you do work with anyone, you should acknowledge your collaborators. Similarly, if you use references such as books, lecture notes, or web pages, you should cite these as well.

2 Corrections

1. In problem 2, I previously wrote δ instead of $1/\delta$.
2. In problem 2, I specified that the diagonal entries of D be positive.
3. In problem 1, I've replaced $\alpha_1(B)$ by $\max_i |\alpha_i(B)|$.
4. In problem 3, I've fixed the multiplier. I mistakenly wrote $\sum_i \frac{1}{w_i}$ when I should have written $\sum w_i$.
5. I've changed the due date to Oct 14, 2009, from Oct 9.

3 Practicing Courant-Fischer

1. Let A and B be symmetric matrices, and for any matrix M let $\alpha_i(M)$ denote the i-th largest eigenvalue of M (so $\alpha_1(M)$ is the largest). Prove that for all i

$$|\alpha_i(A+B) - \alpha_i(A)| \leq \max_i |\alpha_i(B)| .$$

2. Let A be a positive semi-definite symmetric matrix and let D be a diagonal matrix in which every diagonal entry is greater than 0 and st most δ. Prove that for all i,

$$\alpha_i(D^{-1/2} A D^{-1/2}) \geq (1/\delta)\alpha_i(A).$$

4 Graphic Inequalities

3. Let $G = (V, E, w)$ be a weighted star graph having edges (i, n) for $1 \leq i \leq n - 1$ where the weight of edge (i, n) is w_i. Let H consist of a weighted complete graph on vertices $\{1, \ldots, n-1\}$ in which the edge (i, j) has weight $w_i w_j$. The vertex n also appears in H, but is not connected to any other vertex. Prove that

$$\left(\sum_i w_i \right) G \succcurlyeq H.$$

4. Prove that there is some absolute constant c such that for $n \geq 2^d - 1$, $\lambda_2(T_n) \geq 1/cn$, where T_n is the complete binary tree on n vertices. (Hint: use the full power of Lemma 6.5.2. That is, use weighted paths with carefully chosen weights).

5. In the following, we let G be a d-regular connected graph on n vertices.

   a. Prove that there is a constant $c > 0$ such that most pairs of vertices (u, v) must be at distance at least $c \log_d n$ in G.

   b. For every pair of vertices $u \neq v$, let $P(u, v)$ be a path in G from u to v. Prove that there exists an absolute constant $c > 0$ such that there must be some edge $e \in E$ that appears in at least $cn \log_d n/d$ of the paths $P(u, v)$.

   c. For every pair of vertices $u \neq v$, let $P(u, v)$ be a weighted path from u to v that uses edges in G and assigns these edges weights so that the sum of the reciprocals of their weights equals 1. Prove that there exists an absolute constant $c > 0$ such that there must be some edge $e \in E$ for which the sum of the weights assigned to e by all paths $P(u, v)$ that use edge e is at least $cn \log_d^2 n/d$.

   So, for a d-regular graph, we cannot use the technique of Lecture 6 to prove a lower bound on λ_2 that is better than $c \log_d^2 n/d$.

5 Path Graphs

6. Let $c_1, \ldots, c_{n-1} > 0$ and let P be the weighted path graph with Laplacian

$$L_P = \sum_{i=1}^{n-1} c_i L_{(i,i+1)}.$$

   You will show that a test vector can be used to prove a lower bound on λ_2! That is, let v be any vector with no zero entries such that

$$v(1) < v(2) < \cdots < v(n).$$

   Prove that

$$\lambda_2(P) \geq \min_{i : v_i \neq 0} \frac{(L_P v)_i}{v_i}.$$

   Hint: Consider the proof of Theorem 3.1.1.