Spectral Graph Theory **out:** November 9, 2009

Problem Set 4

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1 Homework Policy

You are allowed to discuss the problems in groups of two or three, but you must write up the solutions on your own. If you do work with anyone, you should acknowledge your collaborators. Similarly, if you use references such as books, lecture notes, or web pages, you should cite these as well.

2 Corrections

I was off by 1 on k in part c of Problem 4. Feel free to replace any ks by $(k+1)$ or $(k-1)$ as needed.

Problem 1: Line Graphs

Let $G = (V, E, w)$ be a d-regular graph, and let $\boldsymbol{v}$ be an eigenvector of the Laplacian of G of eigenvalue λ. Let H be the line graph of G. Now, construct the vector $\boldsymbol{w} \in \mathbb{R}^E$ by

$$\boldsymbol{w}(i,j) = \boldsymbol{v}(i) + \boldsymbol{v}(j).$$

a. Prove that $\boldsymbol{w}$ is an eigenvector of H of eigenvalue λ.

b. Show that this does not necessarily hold when G is not a regular graph. In particular, present a graph G and an eigenvector $\boldsymbol{v}$ of G for which this construction produces a graph H for which the vector $\boldsymbol{w}$ is not even an eigenvector.

Problem 2: Removing edges from graphs

Let G be an unweighted graph (just for simplicity) on n vertices. Recall that L_G^+ is the pseudo-inverse of L_G, which was defined in Lecture 16. Prove that if H is a subgraph of G obtained by removing one edge from G, then the matrix

$$L_G^+ L_H$$

has at most 3 distinct eigenvalues.

Hint: use Courant-Fischer.

Problem 3: Growth rates and eigenvalues

Let $G = (V, E)$ be an unweighted d-regular graph. Let $S \subset V$ such that $|S| \leq |V|/4$. Let $N^0(S) = S$, and for $j \geq 1$ let $N^j(S)$ denote the set of vertices at distance exactly j from S. Let

$$B^k(S) = \cup_{j \leq k} N^k(S)$$

denote the set of vertices at distance at most k from S.

Let k be the largest integer such that $\left|B^k(S)\right| < 2|S|$. We will prove that $k < \sqrt{8d/\lambda_2}$, where λ_2 is the second eigenvalue of the Laplacian of G.

a. Define the column vector $\boldsymbol{x}$ by

$$\boldsymbol{x}(u) = \begin{cases} k - i & \text{if } u \in N^i(S), \text{ for } i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\boldsymbol{x}^T L \boldsymbol{x} \leq 2d|S|,$$

where L is the Laplacian matrix of G.

b. Define the vector $\boldsymbol{y}$ by

$$\boldsymbol{y} = \boldsymbol{x} - \frac{1}{n}\mathbf{1}\mathbf{1}^T\boldsymbol{x}.$$

Prove that

$$\boldsymbol{y}^T\boldsymbol{y} \geq \left(\frac{k}{2}\right)^2 |S|.$$

c. Prove that

$$k < \sqrt{\frac{8d}{\lambda_2}}.$$

So, for $k \geq \sqrt{\frac{8d}{\lambda_2}}$, $\left|B^k(S)\right| \geq 2|S|$.

3 Problem 4: Finishing lecture 15

First, download the improved lecture notes for lecture 15.

Recall the matrix

$$S \stackrel{\text{def}}{=} \begin{bmatrix} \omega M & (1-\omega)I \\ I & 0 \end{bmatrix},$$

from Lecture 15, and recall that we have set μ so that all of the eigenvalues of M lie **strictly** between $-\mu$ and μ, and that we have set

$$\omega = \frac{2}{1+\sqrt{1-\mu^2}}.$$

So, for every eigenvector $\boldsymbol{v}$ of M, S contains two eigenvectors $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$. By requiring that all eigenvalues of M lie strictly between $-\mu$ and μ, we ensure that the eigenvalues of $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$ are distinct.

a. For any eigenvector $\boldsymbol{v}$ of M, determine coefficients c^+ and c^- so that

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{pmatrix} = c^+ \boldsymbol{v}^+ + c^- \boldsymbol{v}^-.$$

b. Compute the value of

$$S^k \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{pmatrix}.$$

c. Prove that

$$\left\| S^k \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{pmatrix} \right\| \leq 2(k+1) \left(\frac{\omega\mu}{2} \right)^{k-1}.$$

d. For any eigenvector $\boldsymbol{v}$ of M, determine coefficients c^+ and c^- so that

$$\begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{v} \end{pmatrix} = c^+ \boldsymbol{v}^+ + c^- \boldsymbol{v}^-.$$

e. Compute the value of

$$S^k \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{v} \end{pmatrix}.$$

f. Prove that

$$\left\| S^k \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{v} \end{pmatrix} \right\| \leq 2(k+1) \left(\frac{\omega\mu}{2} \right)^{k}.$$

g. Prove that

$$\left\| S^k \right\| \leq 2(k+1) \left(\frac{\omega\mu}{2} \right)^{k}$$