Spectral Graph Theory **out:** November 19, 2009

Problem Set 5

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1 Homework Policy

You are allowed to discuss the problems in groups of two or three, but you must write up the solutions on your own. If you do work with anyone, you should acknowledge your collaborators. Similarly, if you use references such as books, lecture notes, or web pages, you should cite these as well.

2 Corrections

None yet.

Problem 1: Line Graphs

We use the notation $x \sim y$ to indicate that x and y are neighbors, and $x \not\sim y$ to indicate that x and y are not neighbors.

a. Let $G = (V, E)$ be a k-regular graph with eigenvalues, r, $2r + 2$ and -2, for $r \geq 3$. Prove that G is strongly regular with parameters $\lambda = r$ and $\mu = 2$.

b. Let v be any vertex. Prove that there exist vertices x and y such that $v \sim x$, $v \sim y$ and $x \not\sim y$.

c. Let v, x be any pair of vertices such that $v \sim x$. Let X be the set of common neighbors of v and x. Let

$$Y = \{y \in V - \{v, x\} : y \sim v, y \not\sim x\}.$$

Prove that every $y \in Y$ has at least $\lambda - 1$ neighbors in Y.

d. Let v, x, X and Y be as defined in part c. Prove that for every $y, u \in Y$, y and u must be neighbors. Conclude that every $y \in Y$ has λ neighbors in Y.

e. Let v, x, X and Y be as defined in part c. Prove that every vertex in X has $\lambda - 1$ neighbors in X.

f. Prove that every vertex $v \in V$ is involved in exactly two cliques having $\lambda + 2$ vertices, and that the intersection of these cliques is precisely v.

g. Let v, x, X and Y be as defined in part c. Let z be a node that is not in $\{v\} \cup X \cup Y$. We know that z has two common neighbors with v. Prove that they cannot both be in X. Conclude that one must be in $X \cup \{x\}$ and the other must be in Y.

h. Prove that G is isomorphic to L_{r+2}.