18.405J/6.841J Advanced Complexity Theory	September 17, 1998

Lecture 3

Lecturer: Daniel A. Spielman — *Scribe: Sudipta Sengupta*

1 Some useful theorems

We first mention a few useful theorems discussed in previous lectures.

Theorem 1.1 (Perron's Theorem) *Let G be a $n-$vertex connected graph with adjacency matrix A. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ denote the eigenvalues of A. Then,*

1. *$\exists$ a vector $\bar{x} > 0$ (component-wise), and $\lambda > 0$ such that $A\bar{x} = \lambda\bar{x}$.*
2. *$\lambda \geq |\lambda_i|$ $forall$ i.*
3. *λ has multiplicity 1.*
4. *If $A\bar{y} = \mu\bar{y}$ for $\bar{y} > 0$, then $\bar{y}$ is a multiple of $\bar{x}$ i.e. $\bar{x}$ is the only eigenvector (upto scalar multiples) with all entries the same sign.*

Theorem 1.2 (Courant-Fischer Theorem) *Let A be a real symmetric matrix with spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then,*

$$\lambda_i = \max_{\mathcal{U} \subseteq \mathcal{R}^{\setminus}} \min_{\bar{x} \in \mathcal{U}} \frac{\bar{x}^T A \bar{x}}{\bar{x}^T \bar{x}}$$

where $\mathcal{U}$ is of dimension i.

Theorem 1.3 (Interlacing Theorem) *Let A be a real symmetric matrix and B a principal $(n-1) \times (n-1)$ minor of A. If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = spec(A)$, and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} = spec(B)$, then*

$$\lambda_i \geq \mu_i \geq \lambda_{i+1} \; \forall \; i.$$

Corollary 1.1 *Let A be a $n \times n$ real symmetric matrix and B any $m \times m$ principal minor of A $(m < n)$. Then, $\forall$ α, A has at least as many eigenvalues less (greater) than α as B does.*

Proof. Apply the interlacing theorem $(n-m)$ times, each time reducing the numbers of rows/columns by one, until you reach matrix B. ∎

Definition 1.1 *H is a* **vertex-induced subgraph** *of G if $\exists$ $S \subseteq V(G)$ such that $H = (S, \{(i,j) \in S \times S | (i,j) \in E(G)\})$.*

We can now state the interlacing theorem and its corollary in terms of graphs.

Theorem 1.4 (graph version of Interlacing Theorem) *Let G be a $n-$vertex graph and H a vertex-induced subgraph of G with $(n-1)$ vertices. If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = spec(G)$, and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} = spec(H)$, then*

$$\lambda_i \geq \mu_i \geq \lambda_{i+1} \; \forall \; i.$$

Corollary 1.2 *Let H be a vertex-induced subgraph of G. Then, $\forall \; \alpha$, G has at least as many eigenvalues less (greater) than α as H does.*

2 Independent sets in graphs

Definition 2.1 *S is an* **independent set** *in G if the vertex-induced subgraph of S has no edges.*

Definition 2.2 $\alpha(G) = \max\{|S| \mid S \text{ independent}\}$.

Theorem 2.1 *Let graph G have spectrum $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then,*

$$\alpha(G) \leq \min(|\{\mathrm{i} \mid \lambda_\mathrm{i} \leq 0\}|, |\{\mathrm{i} \mid \lambda_\mathrm{i} \geq 0\}|)$$

Proof. If S is independent, then the vertex-induced subgraph of S has a $|S| \times |S|$ zero matrix as its adjacency matrix. Hence, S has eigenvalue 0 with multiplicity $|S|$. The result now follows from Corollary 1.2. ∎

Theorem 2.2 *For a $d-$regular connected graph G with n vertices and spectrum $\lambda_1(= d) \geq \lambda_2 \geq \ldots \geq \lambda_n$,*

$$\alpha(G) \leq \frac{-n\lambda_n}{d - \lambda_n}$$

Proof. Let J denote the $n \times n$ all $1's$ matrix. Consider the matrix $A - cJ$. If S is an independent set in G, then there is a $|S| \times |S|$ minor of $A - cJ$ in which all entries are $-c$. The eigenvalues of this minor are $-c|S|$ (multiplicity 1), and 0 (multiplicity $|S| - 1$). Hence, from Corollary 1.1, $-c|S| \geq$ minimum eigenvalue of $A - cJ$.

$\bar{1}$ vector is an eigenvector of J with eigenvalue n. It is also an eigenvector of A with eigenvalue $\lambda_1 = d$. Consider any other eigenvector $\bar{x}$ of A with eigenvalue $\lambda \neq d$. Since $A\bar{x} = \lambda\bar{x}$, we have

$$\sum_{\{j:(i,j) \in E(G)\}} x_j = \lambda x_i \; \forall \; i.$$

Summing up these n equations, we have

$$d \sum_{i=1}^{n} x_i = \lambda \sum_{i=1}^{n} x_i$$

which implies $\sum_{i=1}^{n} x_i = 0$ (since $\lambda \neq d$) i.e. $\bar{x}$ is an eigenvector of J with eigenvalue 0. Thus, every eigenvector of A is an eigenvector of J. Hence, the eigenvalues of $A - cJ$ are $\lambda_2, \ldots, \lambda_n, \lambda_1 - cn$ (not necessarily in order).

Using the inequality derived in the first paragraph of the proof, we have $-c|S| \geq \min(\lambda_\mathrm{n}, \lambda_1 - \mathrm{cn})$. To obtain the best bound, we set $\lambda_n = \lambda_1 - cn$, whence $c = \frac{\lambda_1 - \lambda_n}{n} = \frac{d - \lambda_n}{n}$. This gives $|S| \leq \frac{-n\lambda_n}{d - \lambda_n}$. ∎

3 Cuts in graphs

Perron's Theorem states that eigenvectors corresponding to λ_s ($s \geq 2$) do not have all entries of the same sign. This suggests partitioning the vertices of the graph into two sets according to the sign of the value associated with each vertex in the eigenvector, and considering the cut induced by this partition. This idea was originally proposed by Fiedler. In this section, we investigate such "Fiedler cuts" of a graph.

Theorem 3.1 *Let G be a $n-$vertex connected graph with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Let $\bar{x}$ be a vector such that $A\bar{x} \geq \lambda_s \bar{x}$ (component-wise) for some $s \geq 2$. Let $M = \{i | x_i \geq 0\}$. Then, M is non-empty, and the vertex-induced subgraph H on M has at most $s - 1$ connected components.*

Proof. (By contradiction). The fact that M is non-empty follows from Perron's Theorem and that $s \geq 2$. Assume that H has t components, where $t \geq s$. Denote these components by $H_{(1)}, H_{(2)}, \ldots, H_{(t)}$. Let $A_{(1)}, A_{(2)}, \ldots, A_{(t)}$ be their adjacency matrices. We can now write the equation $A\bar{x} \geq \lambda_s \bar{x}$ as:

$$\begin{pmatrix} A_{(1)} & 0 & . & . & 0 & C_{(1)} \\ 0 & A_{(2)} & 0 & . & 0 & C_{(2)} \\ 0 & . & . & 0 & . & . \\ 0 & . & . & . & 0 & . \\ 0 & . & . & 0 & A_{(t)} & C_{(t)} \\ {C_{(1)}}^T & {C_{(2)}}^T & . & . & {C_{(t)}}^T & D \end{pmatrix} \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ . \\ . \\ x^{(t)} \\ x^{(0)} \end{pmatrix} \geq \lambda_s \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ . \\ . \\ x^{(t)} \\ x^{(0)} \end{pmatrix}$$

This gives

$$A_{(i)} x^{(i)} + C_{(i)} x^{(0)} \geq \lambda_s x^{(i)}$$

where, $x^{(i)} \geq 0$ (and $\neq 0$), $C_{(i)} \geq 0$ (and $\neq 0$, since G is connected) $\forall\ i$, and $x^{(0)} < 0$. Hence,

$$A_{(i)} x^{(i)} \geq \lambda_s x^{(i)} \text{ (and } \neq)$$

or, $x^{(i)^T} A_{(i)} x^{(i)} > x^{(i)^T} \lambda_s x^{(i)}$

Using the above inequality and Courant-Fischer theorem, it follows that the greatest eigenvalue of $A_{(i)}$ is greater than λ_s.

We now use the fact that for any $n \times n$ matrix A and $m \times m$ matrix B,

$$spec \begin{pmatrix} A & 0 \\ 0 & b \end{pmatrix} = spec(A) \cup spec(B)$$

where, the union is a multi-set union. This implies that matrix A' given by

$$A' = \begin{pmatrix} A_{(1)} & 0 & . & . & 0 \\ 0 & A_{(2)} & 0 & . & 0 \\ 0 & 0 & . & . & 0 \\ 0 & 0 & 0 & . & . \\ 0 & . & . & 0 & A_{(t)} \end{pmatrix}$$

has atleast t eigenvalues greater than λ_s. If $t \geq s$, this contradicts Corollary 1.1, since A has $(s-1)$ eigenvalues greater than λ_s. ∎

The above theorem generalizes to matrices, but before we state that, let us introduce some matrix terminology. If A is an $n \times n$ matrix and M a proper non-empty subset of $N = \{1, 2, \ldots, n\}$, we shall denote by $A(M)$ that principal submatrix of A whose indices of the rows (and columns) belong to M.

Definition 3.1 *An $n \times n$ matrix A is called* **irreducible** *if for no decomposition of N into two non-empty subsets N_1, N_2, $A_{ij} = 0$ whenever $i \in N_1, j \in N_2$.*

Definition 3.2 *A symmetric $n \times n$ matrix A is of* **degree of reducibility** *s, $0 \leq s \leq n-1$ if there exists a decomposition of $N = \{1, 2, \ldots, n\}$ into $s+1$ non-empty subsets $N = N_1 \cup N_2 \cup \ldots \cup N_{s+1}$ such that*

1. *$A(N_i)$ is irreducible for each i*
2. *$A_{pq} = 0$ whenever $p \in N_i, q \in N_j, i \neq j$*

The connection between the number of connected components of a graph G and the degree of reducibility of its adjacency matrix $A(G)$ is easy to see from Definition 3.2 and is stated in the following lemma.

Lemma 3.1 *Let graph G have s connected components . Then, the degree of reducibility of its adjacency matrix $A(G)$ is $s-1$.*

We now state the generalized version of Theorem 3.1 which applies to any non-negative, irreducible, symmetric matrix. The proof of Theorem 3.1 very easily generalizes to the proof of this theorem.

Theorem 3.2 *Let A be a $n \times n$ non-negative, irreducible, symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Let $\bar{v}$ be a vector such that $A\bar{v} \geq \lambda_s \bar{v}$ (component-wise) for some $s \geq 2$. Let $M = \{i \mid v_i \geq 0\}$. Then, M is non-empty and the degree of reducibility of the matrix $A(M)$ does not exceed $s-2$.*

4 Laplacian matrix of a graph

Definition 4.1 *Let G be a graph with n vertices. The* **Laplacian matrix** *of G (denoted by $L(G)$) is the $n \times n$ matrix with entries given by*

$$L_{ij} = \begin{cases} -1 & \text{if } (i,j) \in E(G) \\ \text{degree}(i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Definition 4.2 *Let G be a graph with n vertices and m edges. The* **Gradient matrix** *of G (denoted by $Q(G)$) is the $n \times m$ matrix with entries given by*

$$Q_{i,(j,k)} = \begin{cases} 1 & \text{if } i = \min(j,k) \text{ and } j \neq k \\ -1 & \text{if } i = \max(j,k) \text{ and } j \neq k \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Here are some interesting properties of the Laplacian matrix.

1. It can be shown that $L = Q^T Q$. Then, $\bar{x}^T L \bar{x} = (\bar{x}^T Q)(Q^T \bar{x})$, and hence, L is *positive semi-definite*, i.e. all eigenvalues of L are non-negative.

2. It is easy to see that the $\bar{1}$ vector is an eigenvector of L with eigenvalue 0. The eigenvalues of L are usually written in ascending order as $\lambda_1 (= 0) \leq \lambda_2 \leq \ldots \leq \lambda_n$.

3. The eigenvalue $\lambda_1 = 0$ has multiplicity equal to the number of connected components of G. Hence, if G is connected, then $\lambda_1 \neq \lambda_2$.

4. For any graph G, if two vertices have the same neighbours, then we can make an (Laplacian) eigenvector by assigning them the values 1 and -1, and putting 0's everywhere else.

5. $\bar{x}^T L \bar{x} = \sum_{(i,j) \in E(G)} (x_i - x_j)^2$

6. If G is $d-$regular, then $L = dI - A$.

Example 1. For the empty graph (with n nodes and no edges), the Laplacian matrix has all 0 entries, and the only eigenvalue is 0 (with multiplicity n). ∎

Example 2. For the complete graph on n vertices, the Laplacian has each diagonal entry equal to $(n-1)$, and each non-diagonal entry equal to -1. The only eigenvalue besides 0 is n (with multiplicity $n-1$). ∎

Example 3. Consider the star graph on n nodes (with one central node and other $n-1$ nodes connected to it). The $\bar{1}$ vector is an eigenvector with eigenvalue 0. We can make $(n-2)$ linearly independent eigenvectors with eigenvalue 1 as follows: take any two "spoke vertices" and assign them values 1 and -1, and put 0's everywhere else. Thus, 1 is an eigenvalue with multiplicity $n-2$. Since we know that the trace of the Laplacian is $2(n-1)$, the remaining eigenvalue must be n. The corresponding eigenvector assigns $-(n-1)$ to the central vertex, and 1 to all the others. ∎

Theorem 4.1 *Let G be a $n-$vertex graph with Laplacian spectrum $\lambda_1(=0) \leq \lambda_2 \leq \ldots \leq \lambda_n$. Let e be an edge not in G. Let the spectrum of the graph $(G+e)$ be $\mu_1(=0) \leq \mu_2 \leq \ldots \leq \mu_n$. Then,*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \ldots \leq \lambda_n \leq \mu_n.$$

Proof. Without any loss of generality, assume that $e = (1,2)$. We then have

$$L(G+e) = L(G) + \begin{pmatrix} 1 & -1 & . & . & 0 \\ -1 & 1 & 0 & . & 0 \\ 0 & . & 0 & . & 0 \\ 0 & . & . & 0 & 0 \\ 0 & 0 & . & . & 0 \end{pmatrix}$$

or, $\bar{x}^T L(G+e)\bar{x} = \bar{x}^T L(G)\bar{x} + (x_1 - x_2)^2$

From this and Courant-Fischer Theorem, it follows that $\lambda_i \leq \mu_i \; \forall \; i$.

The other part of the proof (i.e. showing that $\mu_i \leq \lambda_{i+1}$) uses the fact that $L(G+e) = L(G)+$ (matrix of rank one), and is left as an exercise. ∎

Theorem 4.2 *Let G be a $n-$vertex connected graph with Laplacian eigenvalues $\lambda_1(= 0) \leq \lambda_2 \leq \ldots \leq \lambda_n$. Let $\bar{v}$ be a vector such that $L\bar{v} = \lambda_s \bar{v}$ for some $s \geq 2$. Let $M = \{i | v_i + c \geq 0\}$ for any $c \geq 0$. Then, M is non-empty and the vertex-induced subgraph H on M has at most $s-1$ connected components.*

Proof. Consider the matrix $A = -L + nI$. A is clearly non-negative, irreducible, and symmetric, and has eigenvalues $n - \lambda_1 \geq n - \lambda_2 \geq \ldots \geq n - \lambda_n$. It is easily verified that the vector $\bar{v'} = \bar{v} + c\bar{1}$ satisfies $A\bar{v'} \geq (n - \lambda_s)\bar{v'}$. Hence, Theorem 3.2 implies that the degree of reducibility of $A(M)$ does not exceed $s-2$. Thus, H has at most $s-1$ connected components. ∎