18.409 Applied Extremal Combinatorics	September 22, 1998

Lecture 4

Lecturer: Daniel A. Spielman | *Scribe: Rudi Seitz*

In this lecture, we will assume that $G = (V, E)$ is d-regular connected undirected graph with $|V| = n$. For any $X, Y \subseteq V$, we will let

$$e(X,Y) = |\{(x,y) \in X \times Y \mid xy \in E\}|.$$

Notice that if X and Y are disjoint then $e(X,Y)$ counts the number of edges in G that connect a vertex in X with a vertex in Y. When X and Y are not disjoint, each edge between two nodes in $X \cap Y$ is counted twice.

For a set $X \subseteq V$ we let

$$N(X) = \{y \in V |\ \exists x \in X\ .\ xy \in E\}.$$

We will be interested in the eigenvalues of the adjacency matrix A of G. In particular, we will let λ represent the second largest absolute value of any eigenvalue of A. (Remember that for a d-regular graph, the largest eigenvalue is d.)

If we choose two sets $X, Y \subseteq V$ at random, where $|X| = \alpha n$ and $|Y| = \beta n$, the expected value of $e(X,Y)$ is $\frac{d|X||Y|}{n} = d\alpha\beta n$. Now we will show that if λ is small, then any two sets $X, Y \subseteq V$ must have about as many edges between them as we would expect if the sets were chosen at random.

Theorem 1 *Let $G = (V, E)$ be a d-regular connected graph on n nodes such that every eigenvalue (of the adjacency matrix of G) other than d has absolute value at most λ. Let $X, Y \subseteq V$ with $|X| = \alpha n$ and $|Y| = \beta n$. Then*

$$|e(X,Y) - d\alpha\beta n| \leq \lambda n \sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}.$$

Proof Let A be the adjacency matrix of G, and let x and y be the characteristic vectors of the sets X and Y. Define $v = x - \alpha 1$ and $w = y - \beta 1$ where 1 denotes the all-ones vector. Our proof will consist of two steps. First, we will show that $\langle Av, w \rangle = e(X,Y) - d\alpha\beta n$. Second, we will bound $|\langle Av, w\rangle|$ from above.

Step 1. Simple calculations show that $\langle Ax, y \rangle = e(X,Y)$ and $\langle Ax, \beta 1 \rangle = \langle \alpha A 1, y \rangle = \langle \alpha A 1, \beta 1 \rangle = d\alpha\beta n$. Hence,

$$\begin{aligned} \langle Av, w \rangle &= \langle A(x - \alpha 1), y - \beta 1 \rangle \\ &= \langle Ax, y \rangle - \langle Ax, \beta 1 \rangle - \langle \alpha A 1, y \rangle + \langle \alpha A 1, \beta 1 \rangle \\ &= e(X,Y) - d\alpha\beta n. \end{aligned}$$

Step 2. Remember that for two vectors a, b we have $\langle a, b\rangle \leq \sqrt{\langle a, a\rangle\langle b, b\rangle}$. Observe that $\langle v, v\rangle = (\alpha - \alpha^2)n$ and $\langle w, w\rangle = (\beta - \beta^2)n$. Noticing that v is perpendicular to the eigenvector 1 corresponding to the largest eigenvalue d, we get

$$|\langle Av, w\rangle| \leq \lambda|\langle v, w\rangle| \leq \lambda\sqrt{\langle v, v\rangle\langle w, w\rangle} = \lambda n\sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}.$$

■

Now we will use this result to prove Tanner's Theorem, which states that G has good expansion if λ is small.

Theorem 2 (Tanner) *Let $G = (V, E)$ be a d-regular connected graph on n nodes such that every eigenvalue (of the adjacency matrix of G) other than d has absolute value at most λ. Then for any $X \subseteq V$,*

$$|N(X)| \geq \frac{d^2|X|}{\lambda^2 + (d^2 - \lambda^2)\frac{|X|}{n}}.$$

Proof Let $Y = V - N(X)$. Set α and β so that $\alpha n = |X|$ and $\beta n = |Y|$. Then $|N(X)| = (1 - \beta)n$. Since $e(X, Y) = 0$, the previous theorem tells us that $d\alpha\beta n \leq \lambda n\sqrt{(\alpha - \alpha^2)(\beta - \beta^2)}$. Hence,

$$\begin{aligned}
d^2\alpha^2\beta^2 &\leq \lambda^2(\alpha - \alpha^2)(\beta - \beta^2) \\
d^2\alpha\beta &\leq \lambda^2(1 - \alpha)(1 - \beta) \\
d^2\alpha\beta &\leq \lambda^2(1 - \alpha - \beta + \alpha\beta) \\
d^2\alpha\beta + \lambda^2\beta - \lambda^2\alpha\beta &\leq \lambda^2 - \lambda^2\alpha \\
\beta(\lambda^2 + (d^2 - \lambda^2)\alpha) &\leq \lambda^2 - \lambda^2\alpha \\
\beta &\leq \frac{\lambda^2 - \lambda^2\alpha}{\lambda^2 + (d^2 - \lambda^2)\alpha} \\
1 - \beta &\geq \frac{\lambda^2 + (d^2 - \lambda^2)\alpha - \lambda^2 + \lambda^2\alpha}{\lambda^2 + (d^2 - \lambda^2)\alpha} \\
1 - \beta &\geq \frac{d^2\alpha}{\lambda^2 + (d^2 - \lambda^2)\alpha} \\
|N(X)| = (1 - \beta)n &\geq \frac{d^2|X|}{\lambda^2 + (d^2 - \lambda^2)\frac{|X|}{n}}
\end{aligned}$$

■

In the remaining part of the lecture, we will no longer be concerned with the eigenvalues of the adjacency matrix. Instead we will consider the second smallest eigenvalue λ_2 of the Laplacian matrix of the graph. For any $S \subset V$, we define the *cut ratio* of S by

$$\phi(S) = \frac{e(S, V \setminus S)}{|S| \cdot |V \setminus S|}.$$

The *isoperimetric number* of the graph is defined as the smallest ratio of any cut:

$$\phi = \min_{S \subset V} \phi(S).$$

We can express ϕ as the solution to an integer program:

$$\frac{\phi}{2} = \min_{\substack{x_i \in \{+1,-1\} \\ x_i\text{'s not all equal}}} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i,j\in V}(x_i - x_j)^2}.$$

To see that this program in fact captures ϕ, notice that cuts $S \subset V$ can be placed in correspondence with vectors $x \in \{+1,-1\}^n$. A vector $x \in \{+1,-1\}^n$ represents a cut S such that $i \in S$ iff $x_i = 1$. Now if x represents S, then it is easy to verify that

$$\sum_{(i,j)\in E} (x_i - x_j)^2 = 4 \cdot e(S, S \setminus V).$$

Similarly, we may check that

$$\sum_{i,j\in V} (x_i - x_j)^2 = 2 \cdot 4 \cdot |S||V \setminus S|.$$

Combining these observations with the definition of ϕ, it follows that the minimum of our program is precisely $\frac{\phi}{2}$, as stated above. By relaxing the integer program, we obtain the inequality

$$\frac{\phi}{2} \geq \min_{\substack{x_i \in \mathcal{R} \\ x_i\text{'s not all equal}}} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i,j\in V}(x_i - x_j)^2}.$$

Notice that the value of the fraction above is not changed if we replace x_i with $x_i + a$ for all i. Therefore we may assume without loss of generality that $\sum_i x_i = 0$. (If not, just set each $x_i := x_i - \frac{\sum_j x_j}{n}$.) Now assuming that $\sum_i x_i = 0$, the denominator can be simplified considerably:

$$\sum_{i,j\in V} (x_i - x_j)^2 = 2n \sum_i x_i^2 - 2 \sum_i x_i \sum_j x_j = 2n \sum_i x_i^2.$$

Hence we get

$$\frac{\phi}{2} \geq \min_{\substack{x_i \in \mathcal{R} \\ x_i\text{'s not all equal} \\ \sum_i x_i = 0}} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{2n \sum_i x_i^2}.$$

But by the Courant-Fischer Theorem, we know that this minimum is precisely $\frac{\lambda_2}{2n}$. This gives the following theorem:

Theorem 3 *Let G be a graph with isoperimetric number ϕ and second Laplacian eigenvalue λ_2. Then $\phi \geq \frac{\lambda_2}{n}$.* ■

In the next lecture, we will work with a slightly different definition of the isoperimetric number, which we will denote by ϕ'. We will prove Cheeger's Theorem, which states that

$$\lambda_2 \geq \frac{\phi'^2}{2d}$$

where d is the maximum degree of any vertex in the graph.