18.409 Applied Extremal Combinatorics	September 29, 1998
Lecture 5	
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In this lecture, we will work with a slightly different definition of the isoperimetric number from the one we used in the last lecture. For a graph G, let $e(X, Y)$ be the number of edges with one endpoint in X and the other in Y. Define the *ratio* of a cut $S \subset V$ by

$$\phi(S) = \frac{e(S, V \setminus S)}{\min(|S|, |V \setminus S|)}.$$

Now define the *isoperimetric number* as the smallest ratio of any cut:

$$\phi = \min_{S \subset V} \phi(S).$$

The goal of this lecture will be to prove Cheeger's Inequality:

Theorem 1 *For any d-regular graph G with second smallest Laplacian eigenvalue λ_2, it holds that*

$$\lambda_2 \geq \frac{\phi^2}{2d}.$$

Proof Let L be the Laplacian matrix of G. By the Courant-Fischer Theorem, it suffices to show that for any vector $x \in R^n$ with $\sum_i x_i = 0$,

$$\frac{\langle x, Lx \rangle}{\langle x, x \rangle} \geq \frac{\phi^2}{2d}.$$

So let x be a vector in R^n with $\sum_i x_i = 0$. For simplicity, assume that n is odd and $x_1 \leq x_2 \leq \cdots \leq x_n$. Consider the embedding of G into the real line as determined by the vector x. The main observation we will use later in the proof is that for $i \leq n/2$, at least ϕi edges must cross the point x_i, and for $i \geq n/2$ at least $\phi(n-i)$ edges must cross x_i.

Let $m = (n+1)/2$ and define the vector y by $y_i = x_i - x_m$. We claim that

$$\frac{\langle x, Lx \rangle}{\langle x, x \rangle} \geq \frac{\langle y, Ly \rangle}{\langle y, y \rangle}.$$

Therefore it will suffice to find a lower bound on the Rayleigh quotient of y. To see this claim, remember that

$$\frac{\langle x, Lx \rangle}{\langle x, x \rangle} = \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_i x_i^2}.$$

Since $(x_i - x_j) = (y_i - y_j)$ it is clear that

$$\langle x, Lx\rangle = \sum_{(i,j)\in E} (x_i - x_j)^2 = \sum_{(i,j)\in E} (y_i - y_j)^2 = \langle y, Ly\rangle.$$

It remains only to show that $\langle y, y\rangle \geq \langle x, x\rangle$. We do this as follows:

$$\langle y, y\rangle = \sum_i (x_i - x_m)^2 = \sum_i x_i^2 - 2x_m \sum_i x_i + nx_m^2 = \sum_i x_i^2 + nx_m^2 \geq \sum_i x_i^2 = \langle x, x\rangle.$$

(In the third equality, we used that $\sum_i x_i = 0$.)

We would like to eliminate all edges that cross the point $y_m = 0$. For each such edge (i,j), we will replace it with two edges: (i,m) and (m,j). Let E' denote the new edge set. Let E'_- denote the edges (i,j) in E' with $i,j \leq m$, and let E'_+ denote the remaining edges. Since $(y_j - y_i)^2 \geq (y_j - y_m)^2 + (y_m - y_i)^2$, we have

$$\frac{\sum_{(i,j)\in E}(y_i - y_j)^2}{\sum_i y_i^2} \geq \frac{\sum_{(i,j)\in E'}(y_i - y_j)^2}{\sum_i y_i^2} = \frac{\sum_{(i,j)\in E'_-}(y_i - y_j)^2 + \sum_{(i,j)\in E'_+}(y_i - y_j)^2}{\sum_{i=1}^m y_i^2 + \sum_{i=m}^n y_i^2}.$$

At this point, we recall the inequality

$$\frac{a+b}{c+d} \geq \min(\frac{a}{c}, \frac{b}{d})$$

when a, b, c, d are positive. Combining this with the previous inequality, we see that it suffices to bound the fractions

$$\frac{\sum_{(i,j)\in E'_-}(y_i - y_j)^2}{\sum_{i=1}^m y_i^2}, \qquad \frac{\sum_{(i,j)\in E'_+}(y_i - y_j)^2}{\sum_{i=m}^n y_i^2}$$

seperately. We will show how to bound the first fraction; a bound on the second follows by a similar argument.

First notice that for $a > 0$,

$$(y_i - y_j)^2 \geq \frac{y_i^2 - (1+a)y_j^2}{1 + 1/a}$$

because

$$y_i^2 = (y_i - y_j + y_j)^2 \leq (1 + 1/a)(y_i - y_j)^2 + (1+a)y_j^2.$$

Applying this inequality repeatedly, we get

$$\frac{\sum_{(i,j)\in E'_-}(y_i - y_j)^2}{\sum_{i=1}^m y_i^2} \geq \frac{\sum_{(i,j)\in E'_-} y_i^2 - (1+a)y_j^2}{(1+1/a)\sum_{i=1}^m y_i^2}.$$

Let S_i denote the number of edges that have one endpoint at i and another at some $j > i$. Similarly, let T_i denote the number of edges that have one endpoint at i and another at some $j < i$. Then the quantity on the right above can be written as

$$\frac{\sum_{i\leq m} y_i^2(S_i - (1+a)T_i)}{(1+1/a)\sum_{i=1}^m y_i^2}.$$

We will now find a value of a so that

$$\frac{\sum_{i<m} y_i^2(S_i - (1+a)T_i)}{()1+1/a)\sum_{i=1}^m y_i^2} \geq \phi/2.$$

Since the y_i^2 terms are strictly decreasing, we need only choose a so that

$$\frac{\sum_{i=1}^{k}(S_i - (1+a)T_i)}{k} \geq \phi/2,$$

for all $1 \leq k \leq m$. Notice that $\sum_{i=1}^{k} S_i$ is the number of edges that have at least one endpoint less than y_{k+1}, and $\sum_{i=1}^{k} T_i$ is the number of edges that have both endpoints less than y_{k+1}. Therefore $\sum_{i=1}^{k}(S_i - T_i)$ is the number of edges that cross y_{k+1}. This number must be at least ϕk. Also notice that $\sum_{i=1}^{k} T_i \leq (d-\phi)k/2$, since the maximum degree of any vertex is d. Therefore, it suffices to choose a so that

$$\frac{\phi k - a(d-\phi)k/2}{k} \geq \phi/2.$$

If we let $a = \phi/(d-\phi)$, the above inequality is satisfied, and we get $(1+1/a) = d/\phi$. This gives the lower bound

$$\frac{\sum_{i<m} y_i^2(S_i - (1+a)T_i)}{(1+1/a)\sum_{i=1}^{m} y_i^2} \geq \frac{\phi^2}{2d},$$

completing the proof. ■