18.409 Applied Extremal Combinatorics	October 1, 1998
Lecture 6	
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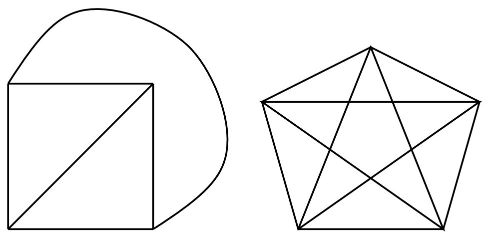
Theorem 1 *Let G be a planar graph of maximum degree d. Let λ_2 be the second smallest eigenvalue of its Laplacian. Then*

$$\lambda_2 \leq \frac{8d}{n}$$

Definition 2 *A graph is planar if there exists a map f from its vertices and edges into the plane (the image of an edge (i, j) is a continuous curve from $f(i)$ to $f(f)$) such that the images of no two edges intersect.*

Example K_4 is planar, K_5 is not, as no matter how we draw it in the plane, two edges must cross.



Theorem 3 (Koebe's embedding theorem) *For every planar graph, there is a set of interior disjoint discs D_i in the plane such that*

$$D_i \text{ touches } D_j \iff (i, j) \in E$$

Theorem 4 (Embedding lemma) $\lambda_2 = \min_{\sum \boldsymbol{v_i}=0} \frac{\sum_{(i,j)\in E} \|\boldsymbol{v_i} - \boldsymbol{v_j}\|^2}{\sum_i \|\boldsymbol{v_i}\|^2}$

Proof First we show that $\lambda_2 \leq RHS$. This is trivial, for if $\boldsymbol{x}$ is an eigenvector of λ_2, simply set $\boldsymbol{v_i} = x_i$.
Next we show that $\lambda_2 \geq RHS$. Let $\boldsymbol{v_i} = (x_i, y_i, \ldots, z_i)$. Then

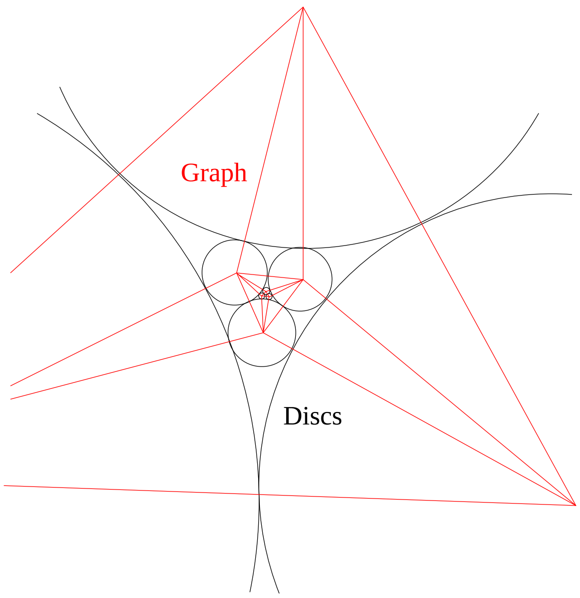
$$
\begin{aligned}
\|\boldsymbol{v_i} - \boldsymbol{v_j}\|^2 &= (x_i - x_j)^2 + (y_y - y_j)^2 + \cdots + (z_i - z_j)^2 \\
\|\boldsymbol{v_i}\|^2 &= x_i^2 + y_i^2 + \cdots + z_i^2 \\
RHS &= \frac{\sum_{(i,j)\in E}(x_i - x_j)^2 + \sum_{(i,j)\in E}(y_i - y_j)^2 + \cdots + \sum_{(i,j)\in E}(z_i - z_j)^2}{\sum_i x_i^2 + \sum_i y_i^2 + \cdots + \sum_i z_i^2}
\end{aligned}
$$

Now we use our favourite inequality $\frac{A+B+\cdots+C}{A'+B'+\cdots+C'} \geq \min(\frac{A}{A'}, \frac{B}{B'}, \cdots, \frac{C}{C'})$ to see that each term in our expression for RHS is exactly our previous formula for λ_2. ■

Proof The obvious plan for our proof is to first use the Koebe theorem to embed our graph in the plane, and then apply the embedding lemma. So for a planar graph G, let $D_1, \ldots, D_n$ be the "kissing disk" embedding given by the Koebe Theorem. Translate the discs in the plane until the centre of gravity of their centres is the origin. Let $\boldsymbol{v_i}$ be the centre of disk D_i. Let d_i be the degree of node i. Let r_i be the radius of disc D_i. Now we can bound $\|\boldsymbol{v_i} - \boldsymbol{v_j}\|^2 = (r_i + r_j)^2 \leq 2r_i^2 + 2r_j^2$. So we can express

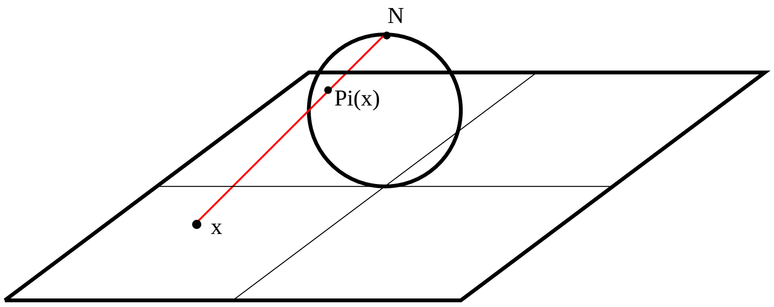
$$\lambda_2 = \min \frac{\sum 2d_i r_i^2}{\sum \|\boldsymbol{v_i}\|^2}$$

But the denominator of this fraction is hard to bound. For example, consider the kissing disk embedding



This graph has three “levels”. If we consider the graph with for example ten levels, we see that the lengths of the edges decrease quickly. So the ratio $\min \frac{\sum 2d_i r_i^2}{\sum \|\boldsymbol{v_i}\|^2}$ for λ_2 approaches a constant, instead of $\frac{1}{\sqrt{n}}$. We need to define a way to alter our kissing disk embedding.

Definition 5 *We define stereographic projection $\Pi : \mathbb{R}^2 \to S$. Draw a sphere above and tangent to $\mathbb{R}^2$ at the origin. Let $\boldsymbol{N}$ be the north pole, ie the top of the sphere. Define $\Pi(\boldsymbol{x})$ to be the intersection of the line segment $[x, \boldsymbol{N}]$ with the sphere.*



We see that circles in $\mathbb{R}^2$ map to spherical caps on the sphere. Use this stereographic projection to get a kissing desk embedding on S. Assume now that the centre of S is at the origin. Let $\boldsymbol{v_i}$ be the centre of the cap D_i. Assume $\sum \boldsymbol{v_i} = \boldsymbol{0}$. Now on S, $\|\boldsymbol{v_i}\|^2 = 1\ \forall i$. Let r_i be the straight line distance from $\boldsymbol{v_i}$ to the edge of D_i. Now $\|\boldsymbol{v_i} - \boldsymbol{v_j}\|^2 \leq (r_i + r_j)^2 \leq 2r_i^2 + 2r_j^2$. So $\sum_{(i,j)\in E} \|\boldsymbol{v_i} - \boldsymbol{v_j}\|^2 \leq 2d \sum r_i^2$. However, the caps are disjoint, so

$$\sum \pi r_i^2 \leq \sum \text{ surface area of the caps } \leq 4\pi$$

So $2d \sum r_i^2 \leq 8d$, and $\lambda_2 \leq \frac{8d}{n}$ as required. ■

However we must justify the assumption that $\sum \boldsymbol{v_i} = \boldsymbol{0}$ (!). We must prove that if $c_1, \ldots, c_n$ are interior disjoint caps on the sphere there exists a cap preserving continuous map $f_{\boldsymbol{\alpha}}$ on the sphere such that the centre of gravity of the centres of $(f_{\boldsymbol{\alpha}}(c_1), \ldots, f_{\boldsymbol{\alpha}}(c_n))$ is the origin. Instead we will prove

Lemma 6 *Let $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ be points on the sphere. Then there exists a cap preserving continuous map $f_{\boldsymbol{\alpha}}$ such that $\sum f_{\boldsymbol{\alpha}}(\boldsymbol{v_i}) = \boldsymbol{0}$*

Proof We sketch the proof. First we define the maps. Remember that S is the unit sphere centered at the origin. Define $\Pi_{\boldsymbol{\alpha}}$ to be stereographic projection onto the plane tangent to the sphere at $\boldsymbol{\alpha}$. Define $\Pi_{\boldsymbol{\alpha}}^{-1}$ to be its inverse. Define $D_{\boldsymbol{\alpha}}^{a}$ to be dilation of the plane tangent to the sphere at $\boldsymbol{\alpha}$ by a (where dilation in $\mathbb{R}^2$ is defined by $(x, y) \mapsto (ax, ay)$). Now the cap preserving continuous map in the lemma is going to be

$$f_{\boldsymbol{\alpha}}(z) = \Pi_{\frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|}}[D_{\frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|}}^{1-\|\boldsymbol{\alpha}\|}(\Pi_{\frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|}}^{-1}(z))]$$

for some $\boldsymbol{\alpha}$ st $\|\boldsymbol{\alpha}\| < 1$.

If $\|\boldsymbol{\alpha}\| = 1$, set $f_{\boldsymbol{\alpha}}(z) = \boldsymbol{\alpha}$, with the one exception $f_{\boldsymbol{\alpha}}(-\boldsymbol{\alpha}) = -\boldsymbol{\alpha}$. Set $\phi(\boldsymbol{\alpha}) = \frac{\sum w(v_i, \boldsymbol{\alpha}) f_{\boldsymbol{\alpha}}(v_i)}{n}$ where w is some weight function.

Now note that if $\boldsymbol{\alpha}$ is near the surface of the sphere, $\phi(\boldsymbol{\alpha})$ is near $\boldsymbol{\alpha}$!

The rest of the proof is easy - we must choose the weight function w so that ϕ is continous, then we will apply some theorems to show that ϕ must have a zero, which proves the lemma.

So set

$$w(v_i, \boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \operatorname{dist}(v_i, \boldsymbol{\alpha}) < 2 - \epsilon \\ (2 - \operatorname{dist}(v_i, \boldsymbol{\alpha}))/\epsilon & \text{otherwise} \end{cases}$$

So ϕ is now continous. Now we apply the following

Lemma 7 *If $\phi : B \to B$ is continuous and ϕ is the identity on $S = \delta B$, then $\exists$ $\boldsymbol{\alpha}$ st $\phi(\boldsymbol{\alpha}) = \mathbf{0}$.*

Proof Consider $b(\phi(\boldsymbol{\alpha}))$ where $b(z) = \frac{z}{\|z\|}$. Now b is a continous function except at $\mathbf{0}$. Assume by way of contradiction that there is no $\boldsymbol{\alpha}$ st $\phi(\boldsymbol{\alpha}) = \mathbf{0}$. Now $b(\phi(\boldsymbol{\alpha})) : b \to B$ is a continuous function. If we consider $-b(\phi(\boldsymbol{\alpha}))$, this is also a continous function, but this function **has no fixed point**. This contradicts the famous

Theorem 8 (Brouwer's fixed point theorem) *If $\phi : B \to B$ is continuous then ϕ has a fixed point.*

which we take without proof. This contradiction proves the lemma. ∎

Now we use this lemma to prove existence of a zero of ϕ. The only remaining thing to show is that we can take $w \equiv 1$, which is true upon taking $\lim\limits_{\epsilon \to 0}$. So we have proved existence of the map $f_{\boldsymbol{\alpha}}$ such that the centre of gravity of the images of the caps is at the origin. $\blacksquare$