18.409 Applied Extremal Combinatorics	October 6, 1998
Lecture 7	
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The aim of this lecture is to develop another method to approximate the isoperimetric number. Recall that the isoperimetric number is defined by

$$\Phi = \operatorname*{Min}_{x_i \in \pm 1} \frac{\sum_{ij\in E}(x_i - x_j)^2}{\sum_{i<j}(x_i - x_j)^2} \tag{1}$$

We have seen one relaxation of this

$$\Phi' = \operatorname*{Min}_{\substack{\mathbf{x_i}\in\mathbb{R}^n,\ \sum \mathbf{x_i}=0 \\ \mathbf{x_i} \text{ not all equal}}} \frac{\sum_{ij\in E}\|\mathbf{x_i} - \mathbf{x_j}\|_2^2}{\sum_{i<j}\|\mathbf{x_i} - \mathbf{x_j}\|_2^2}$$

Here is another

$$\Phi'' = \operatorname*{Min}_{\substack{\mathbf{x_i}\in\mathbb{R}^n \\ \mathbf{x_i} \text{ not all equal}}} \frac{\sum_{ij\in E}\|\mathbf{x_i} - \mathbf{x_j}\|_1}{\sum_{i<j}\|\mathbf{x_i} - \mathbf{x_j}\|_1} \tag{2}$$

In fact it turns out that Φ'' is not a relaxation at all.

Theorem 1 $\Phi = \Phi''$

Proof We will show first that the $\mathbf{x_i}$ in the formula for Φ'' could be one-dimensional and secondly that we can restrict their value to $\{-1, +1\}$.

1. Let $\mathbf{x_i} = (x_{i1}, x_{i2}, \ldots, x_{in})$. Then

   $$\frac{\sum_{ij\in E}\|\mathbf{x_i} - \mathbf{x_j}\|_1}{\sum_{i<j}\|\mathbf{x_i} - \mathbf{x_j}\|_1} = \frac{\sum_r \sum_{ij\in E}|x_{ir} - x_{jr}|}{\sum_r \sum_{i<j}|x_{ir} - x_{jr}|} \geq \operatorname*{Min}_r \frac{\sum_{ij\in E}|x_{ir} - x_{jr}|}{\sum_{i<j}|x_{ir} - x_{jr}|}$$

   and since we are looking for a minimum it is clear that we will find this in one-dimensional space i.e. we may consider just $x_i \in \mathbb{R}$.

2. We will show that the x_i need take on only two values. That being the case it is clear that these two values may as well be ± 1. So assume the x_i take on more than two distinct values. In addition, assume that $s < t < u$ are three such consecutive values. Consider the effect of varying t within $[s, u]$. Now, for $x_i \leq s$ or $x_i \geq u$, $|x_i - t|$ is linear over the range $[s, u]$. Hence

   $$\Phi'' = \operatorname*{Min}_{x_i\in\mathbb{R}} \frac{\sum_{ij\in E}|x_{ir} - x_{jr}|}{\sum_{i<j}|x_{ir} - x_{jr}|}$$

is a ratio of linear functions in t. Thus, it is minimised at some extreme point, i.e. $t \in \{s, u\}$. By induction it follows that the x_i need take on only two distinct values.

The result then follows as $(x_i - x_j)^2 = |x_i - x_j|$ for $x \in \{-1, +1\}$. ■

We can't compute Φ'' in polynomial time, so we consider the following linear program relaxation

$$\operatorname*{Min}_{\text{metric } d} \frac{\sum_{ij \in E} d_{ij}}{\sum_{i<j} d_{ij}}$$

Imposing the normalisation $\sum_{i<j} d_{ij} = 1$ we have

$$\begin{aligned} \operatorname*{Min}_{\text{metric } d} \sum_{ij \in E} d_{ij} \quad \text{s.t.} \quad & d_{ij} \geq 0 \ \ \forall i \\ & \sum_{i<j} d_{ij} = 1 \\ & d_{ij} + d_{jk} \geq d_{ik} \ \ \forall i, j, k \end{aligned} \tag{3}$$

This relaxation can be solved in polynomial time and we will prove that it gives a solution within a logarithmic factor of Φ''. To prove this, we show that any metric can be embedded in an l_1 metric with at most with O(log n) distortion, efficiently.

Theorem 2 *Let d be a metric on n points. Then we can find, in polynomial time, vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in \mathbb{R}^n$ satisfying $d_{ij} \geq \|\mathbf{v_i} - \mathbf{v_j}\|_1 \geq \frac{1}{\mathrm{O}(\log n)}\ d_{ij} \ \ \forall i, j$.*

Proof If $A \subseteq [n]$, then define $d_i(A) = Min_{j \in A}\ d_{ij}$. Now consider the following randomised embedding. Choose at random the sets $A_{a,b} \subseteq [n]$, each of size 2^b, for $1 \leq a \leq \mathrm{O}(\log n)$ and $0 \leq b \leq \lfloor \log n \rfloor$. Hence the vector $\mathbf{v_i} = (d_i(A_{a,b}))_{a,b}$ is of size $\mathrm{O}(\log^2 n)$. Consider the sets $B^c(i, \rho) = \{j \in [n] : d_{ij} \leq \rho\}$ and $B^o(i, \rho) = \{j \in [n] : d_{ij} < \rho\}$, where c and o refer to closed and open, respectively.

Notice that

$$\begin{aligned} A_1 \cap B^o(i, \rho_2) = \emptyset \text{ and } A_1 \cap B^c(j, \rho_1) \neq \emptyset &\Rightarrow d_i(A_1) \geq \rho_2 \text{ and } d_j(A_1) \leq \rho_1 \\ &\Rightarrow |d_i(A_1) - d_j(A_1)| \geq \rho_2 - \rho_1 \end{aligned}$$

So, given sets $A_1, A_2, \ldots, A_t$ we have

$$\begin{aligned} A_k \cap B^o(i, \rho_{k+1}) = \emptyset \text{ and } A_k \cap B^c(j, \rho_k) \neq \emptyset &\Rightarrow |d_i(A_k) - d_j(A_k)| \geq \rho_{k+1} - \rho_k \\ &\Rightarrow \sum_{k=1}^{t} |d_i(A_k) - d_j(A_k)| \geq \rho_{t+1} - \rho_1 \end{aligned}$$

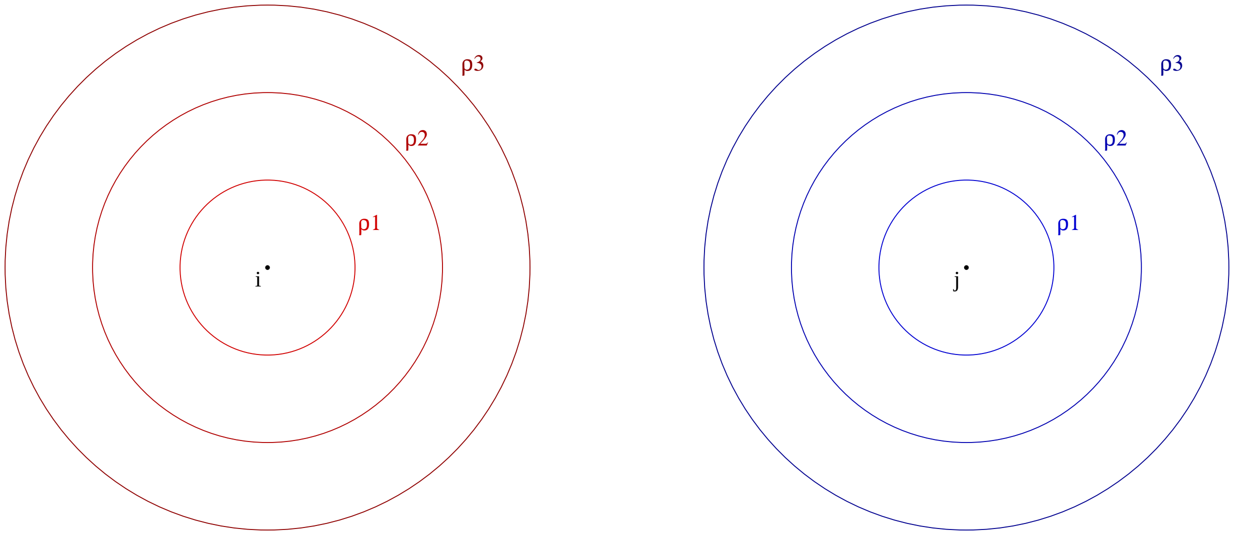

Figure 1: Distances

The idea is to use these sets for the embedding leading to an approximation of the d_{ij}. The first direction is clear, $\|\mathbf{v_i} - \mathbf{v_j}\|_1 \leq \mathrm{O}(\log^2 n)\ d_{ij}$. This follows form the fact that $\mathbf{v_i} = (d_i(A_{a,b}))_{a,b}$ implies that

$$\begin{aligned}\|\mathbf{v_i} - \mathbf{v_j}\|_1 &= \sum_{a,b} |d_i(A_{a,b}) - d_j(A_{a,b})| \\ &\leq \sum_{a,b} d_{ij} \\ &\leq \mathrm{O}(\log^2 n)\ d_{ij}\end{aligned}$$

For the opposite direction, fix i and j. We will show that $\|\mathbf{v_i} - \mathbf{v_j}\|_1 \geq \Omega(\log n)\ d_{ij}$ with probability at least $1 - \frac{1}{2n^2}$. It follows immediately that the result holds for all i and j with probability at least $\frac{1}{2}$.

Let $\rho_k < \frac{d_{ij}}{4}$ be minimal s.t. $|B^c(i, \rho_k)| \geq 2^k$ and $|B^c(j, \rho_k)| \geq 2^k$. In addition, let t be the largest index satisfying $\rho_t < \frac{d_{ij}}{4}$. We then define $\rho_{t+1} = \frac{d_{ij}}{4}$ to be our boundary condition. Note that $B^c(i, \rho_l)$ and $B^c(j, \rho_m)$ are disjoint. Consider fixing k.

Assume that $|B^c(i, \rho_k)| < 2^k$ (otherwise we can repeat the argument for j). On the other hand $|B^o(j, \rho_{k-1})| \geq 2^{k-1}$. So a random set of size $\Theta(\frac{n}{2^k})$ has a constant probability of intersecting $B^o(j, \rho_{k-1})$ but missing $B^c(i, \rho_k)$. Choosing b s.t. $\frac{n}{2^{b+1}} < 2^k < \frac{n}{2^b}$ we have

$$\Pr\left[|d_i(A_{a,b}) - d_j(A_{a,b})| \geq \rho_k - \rho_{k-1}\right] = \text{constant}$$

Now consider varying the a's. We have $a = 1, 2, \ldots, s \log n$. So we can use the Chernoff bound to pick an s s.t.

$$\Pr\left[\sum_{a} |d_i(A_{a,b}) - d_j(A_{a,b})| < \varepsilon \log n(\rho_k - \rho_{k-1})\right] < \frac{1}{2n^3}$$

It follows that

$$\Pr\left[\sum_{a,b} |d_i(A_{a,b}) - d_j(A_{a,b})| < \varepsilon \log n(\rho_{t+1} - \rho_0)\right] < \frac{1}{2n^2}$$

Hence, with probability $1 - \frac{1}{2n^2}$, we have $\|\mathbf{v_i} - \mathbf{v_j}\|_1 \geq \varepsilon \log n(\rho_{t+1} - \rho_0) = \frac{d_{ij}}{4} \varepsilon \log n$. The theorem then follows after scaling by a factor of $\mathrm{O}(\log^2 n)$. ■