18.409 Applied Extremal Combinatorics	October 8, 1998
Lecture 8	
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In the last class, we introduced the linear programming relaxation that approximates the isoperimetric number of a graph G. We proved that this approximation is within $O(\log n)$. In today's class, we will see some examples that demonstrate when the linear programming approximation coincides with the isoperimetric number, when it differs by $O(\log n)$, and what kinds of cuts it induces in different graphs.

1 Review of the LP Approximation

Recall that in the last lecture, we have examined the following quantity, given a graph $G = (V, E)$:

$$L(G) = \min_{\text{metric } d_{ij}} \frac{\sum_{(i,j)\in E(G)} d_{ij}}{\sum_{i<j} d_{ij}}$$

Recall that we required that the metric be such that $\sum d_{ij} > 0$ and $\forall i, j,\ d_{i,j} \geq 0$. We also showed that there exists a metric that achieves $L(C)$ that has only 0/1 values. Note that we may restrict the class of metrics to consider. Suppose the edge weights d_{ij} for $(i, j) \in E$ are given. Then for any $(i, j) \notin E$, the edge-weight induced metric is obtained by setting

$$d_{ij} = \min_{p \text{ a path from } i \text{ to } j} \sum_{(a,b)\in p} d_{ab}$$

This is a natural class of metrics.

Claim 1 *The metric that achieves $L(G)$ is an edge-weight induced metric.*

Proof Suppose the metric $\{d_{ij}\}$ that achieves $L(G)$ is given. Then, since it is a metric, it must satisfy the constraint that $d_{ij} \leq \sum_{(ab)\in p} d_{ab}$ for any p that is a path from i to j. Suppose that for some $d_{ij} \notin E$, $d_{ij} < \sum_{(ab)\in p} d_{ab}$ for all paths p from i to j. Then $L(G)$ is not minimized by $\{d_{ij}\}$, since the denominator of $L(G)$ can be increased without affecting the numerator, by adding

$$\min_{p \text{ a path from } i \text{ to } j} \sum_{(a,b)\in p} d_{ab} - d_{ij}$$

to d_{ij}. This is a contradiction. ■

Now we are going to look at some graphs and see how well $L(G)$ approximates the isoperimetric number $\Phi(G) = min_{S\subset V} \frac{E(S,S^c)}{|S||S^c|}$.

2 Path Graph

Consider the graph in Figure 1. Suppose the edge weights are, as in the figure, a, b, c, d, and e. Our goal is to find a 0/1 assignment to the edge weights that minimizes $L(G) = \frac{a+b+c+d+e}{5a+8b+9c+8d+5e}$. It is straightforward to verify that this expression is minimized when $a = b = c = d = 0$, and $c = 1$. The value we obtain is $\frac{1}{9}$, which is exactly the isoperimetric number for a six-vertex path graph, because:

$$\Phi(G) = \min_{S\subset V} \frac{E(S, S^c)}{|S||S^c|} = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$$

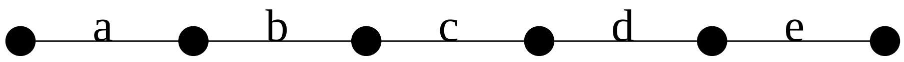

Figure 1: The path graph

Also observe that the edge to which this minimization assigned a non-zero value is the one that gives the optimal cut. Therefore, for a path graph, the linear programming approximation gives the right answer.

It turns out that the LP approximation gives the exact solution for trees in general. We will not formally prove this, but intuitively it follows because in a tree any pair of vertices is connected by a unique path. Also, any cut in the tree cuts just one edge. Cutting the edge that most paths go through should then give the best cut, and this is exactly what the LP approximation will suggest.

3 Hypercube

A hypercube of dimension d is a graph on 2^d nodes, each labeled with its own d-bit string. Two nodes are connected by an edge iff their labels differ in one bit.

Definition 2 *A permutation* π *is an* automorphism *of a graph* G *if*

1. π *is a permutation on* $V(G)$.
2. $(\pi(i), \pi(j)) \in E(G) \Leftrightarrow (i, j) \in E(G)$.

Suppose G is a hypercube graph, π is an automorphism of G and the metric $\{d_{ij}\}_{i,j}$ that achieves $L(G)$ is given. Consider the metric $\{d'_{ij}\}_{i,j} = \{d_{\pi(i),\pi(j)}\}_{i,j}$. Since π is an automorphism of G, d' achieves $L(G)$ as well. Suppose the metrics $\{d_{ij}\}_{i,j}$ and $\{d'_{ij}\}_{i,j}$ minimize $L(G)$. Consider the metric $\{\tilde{d}_{ij}\}_{i,j} = \{d'_{ij} + d_{ij}\}_{i,j}$. It achieves $L(G)$ also, because

$$\frac{\sum_{(i,j)\in E(G)} \tilde{d}_{ij}}{\sum_{i<j} \tilde{d}_{ij}} = \frac{\sum_{(i,j)\in E(G)} d_{ij} + \sum_{(i,j)\in E(G)} d'_{ij}}{\sum_{i<j} d_{ij} + \sum_{i<j} d'_{ij}} \leq \max\left(\frac{\sum_{(i,j)\in E(G)} d_{ij}}{\sum_{i<j} d_{ij}}, \frac{\sum_{(i,j)\in E(G)} d'_{ij}}{\sum_{i<j} d'_{ij}}\right) = L(G)$$

(Recall that $\frac{A+B}{C+D} \leq \max(\frac{A}{B}, \frac{C}{D})$.) Then, by the same derivation, the metric

$$\{\check{d}_{ij}\}_{i,j} = \{\sum_{\pi\in \mathrm{Aut}(G)} d_{\pi(i),\pi(j)}\}_{i,j}$$

also achieves $L(G)$. But, by the definition of the automorphism group of a graph, for any pair of edges (i_1, j_1) and (i_2, j_2) of a hypercube, there exists an automorphism π such that $(\pi(i_1), \pi(j_1)) = (i_2, j_2)$. Therefore, this metric assigns the same weight to each edge of G. Assume without loss of generality that this value is 1. With this observation, we can now compute the value of $L(G)$.

$$L(G) = \frac{\sum_{(i,j)\in E} 1}{(1/2)\sum_{i\neq j; i,j\in\{0,1\}^d} d_{ij}} = \frac{2d2^{d-1}}{2^d \sum_{j\neq \bar{0}} d_{j,\bar{0}}} = \frac{2d2^{d-1}}{2^d \sum_{k=1}^{d} k\binom{d}{k}} = \frac{2d2^{d-1}}{2^d d 2^{d-1}} = \frac{1}{2^{d-1}}$$

It turns out that this bound is tight. First recall that $L(G) \leq \Phi(G)$, because $L(G)$ is a relaxation. Now, since for any cut that cuts a graph into S and S^c it is true that $\Phi(G) \leq \frac{E(S,S^c)}{|S||S^c|}$. Consider the cut that cuts the hypercube of dimension d into two hypercubes of dimension $d-1$. Then $\Phi(G) \leq \frac{2^{d-1}}{2^{d-1}2^{d-1}} = \frac{1}{2^{d-1}} = L(G)$. So we get $L(G) \leq \Phi(G) \leq L(G)$, and therefore $L(G) = \Phi(G)$ if G is a hypercube.

Now, what does the metric that achieves $L(G)$ tell us about the optimal cut? Since it assigns the same weight to each edge, it tells us that any edge is part of some optimal cut. Even though it does not exactly spell out where the cut is, it still tells us something correct about the optimal cut.

4 Expander Graph

For some graphs, the linear programming relaxation for the isoperimetric number may be off by $O(\log n)$. Expander graphs are examples of such graphs. Let us first define expander graphs. There are many ways of doing so, we will see later in the semester that they are roughly equivalent. The following definition of expanders captures the essence of this notion for our present purposes:

Definition 3 *$\mathcal{F}$ is an infinite family of expander graphs if there exist constants d, α such that for all $G \in \mathcal{F}$, the maxdeg$(G) \leq d$ and $\Phi(G) \geq \frac{\alpha d}{n}$.*

It is easy to show by probabilistic techniques that such families exist, but it is tricky to find explicit constructions. We will talk about them later in the semester.

Now let us analyze how close to the isoperimetric number we get when we try to approximate it using the LP relaxation. Suppose G is a graph from an infinite family of expander graphs with maximum degree d and expansion α. Consider the following metric: $d_{i,j} = 1$ for all $(i, j) \in E(G)$. Since $L(G)$ is the minimum over all metrics,

$$L(G) \leq \frac{\sum_{(i,j)\in E(G)} d_{i,j}}{\sum_{i<j} d_{i,j}}$$

Now, since the degree of any node in G is at most d, there are at most $dn/2$ edges. That gives us an upper bound for the numerator. For the denominator, we have the lower bound of $n\frac{n}{2}\log_d\lfloor\frac{n}{2}\rfloor$, because for any node i of G, there are at most d^k nodes of distance $\leq k$ from i. That implies that most nodes are at distance $O(\log n)$ from i, more precisely, at least at distance $\log_d\lfloor\frac{n}{2}\rfloor$. So we have:

$$L(G) \leq \frac{\sum_{(i,j)\in E(G)} d_{i,j}}{\sum_{i<j} d_{i,j}} \leq \frac{dn/2}{n\frac{n}{2}\log_d\lfloor\frac{n}{2}\rfloor} = O(\frac{1}{n\log n})$$

Now we can see that $L(G)$ differs from $\Phi(G)$ by a factor of at least $O(\log n)$, just from the definition of expander graphs. And we know from last class that it cannot differ from it by a bigger factor than $O(\log n)$, so the expander graph is an example of a case when the LP relaxation is as far from the the isoperimetric number as can be.

This analysis tells us that the LP relaxation does not give a good approximation for the isoperimetric number in this particular case. But what if it still induces the optimal cut, i.e. if S is the cut suggested by the LP relaxation, what if $\frac{|E(S,S^c)|}{|S||S^c|}$ still coincides with $\Phi(G)$? The thing is that all the cuts are about the same in expanders (will see later in the semester), and so the cut that the LP relaxation suggests is about as good as the optimal. Are there any graphs where the LP relaxation is not only off by a logarithmic factor, but also suggests a cut that is far from the optimal cut by a logarithmic factor? As we will see from the next example, the answer is yes.

5 Product of an Expander and a Path

Recall the definition of the product graph $G \times H$:

Definition 4 *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. Then their Cartesian product is the graph*

$$G \times H = ((V_G \times V_H), \{((a_1, b), (a_2, b)) : (a_1, a_2) \in E_G, ((a, b_1), (a, b_2)) : (b_1, b_2) \in E_H\})$$

What does the product graph of an expander and a path graph look like? Suppose the expander graph G is as in Figure 2, and the path graph H is as in Figure 1. Then their product graph is as in Figure 3. We will see that, while the optimal cut for this graph is, similar to the cut for the path,

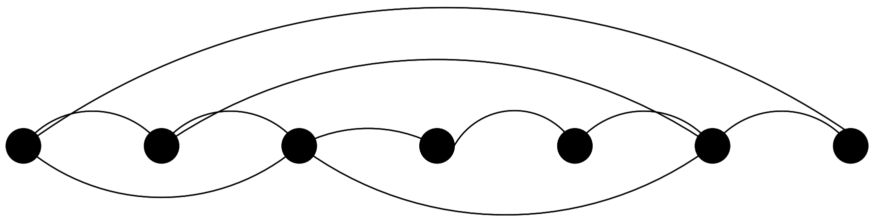

Figure 2: An expander graph

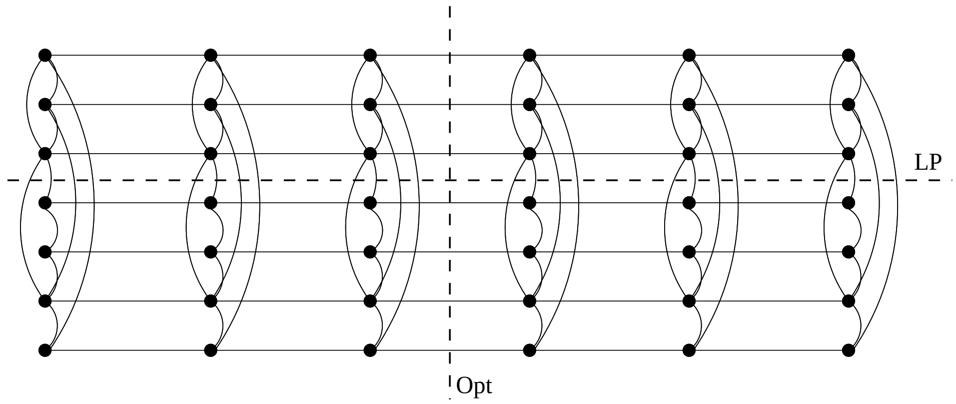

Figure 3: The product graph of the path and the expander

down the middle (see the figure), the LP relaxation will suggest a cut that cuts only the edges that come from the expander, and, as a result, the cut suggested by the LP relaxation is far from the optimal one.

We begin by proving the following lemma:

Lemma 5 *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs such that $L(G) = \alpha$ and $L(H) = \beta$. Then*

$$L(G \times H) = \min(\frac{\alpha}{|V_H|}, \frac{\beta}{|V_G|}).$$

Also, if $\frac{\alpha}{|V_H|} < \frac{\beta}{|V_G|}$, then the metric that achieves $L(G \times H)$ assigns weight 0 to all the edges of the form $((a, b_1), (a, b_2))$ such that $(b_1, b_2) \in E_H$.

Proof First, we will show that

$$L(G \times H) \leq \min(\frac{\alpha}{|V_H|}, \frac{\beta}{|V_G|})$$

Assume without loss of generality that $\frac{\alpha}{|V_H|} \leq \frac{\beta}{|V_G|}$. Suppose d^G is a metric for graph G that achieves $L(G)$. Let an edge weight induced metric d for graph $G \times H$ be such that $\forall (b_1, b_2) \in E_H$, $d_{(a,b_1),(a,b_2)} = 0$, and $\forall (a_1, a_2) \in, E_G d_{(a_1,b),(a_2,b)} = d^G_{a_1,a_2}$. Now we see that

$$L(G \times H) \leq \frac{2 \sum_{((a_1,b_1),(a_2,b_2)) \in E} d_{(a_1,b_1),(a_2,b_2)}}{\sum_{(a_1,b_1) \neq (a_2,b_2)} d_{(a_1,b_1),(a_2,b_2)}} = \frac{2|V_H| \sum_{(i,j) \in E_G} d^G_{ij}}{\sum_{b_1} \sum_{b_2} \sum_{(a_1,a_2)} d^G_{a_1,a_2}} = \frac{\alpha}{|V_H|}$$

Now we have to show that

$$L(G \times H) \geq \min(\frac{\alpha}{|V_H|}, \frac{\beta}{|V_G|})$$

Suppose a metric $d_{(a_1,b_1),(a_2,b_2)}$ that achieves $L(G \times H)$ is given. Let

$$g^b_{a_1,a_2} = d_{((a_1,b),(a_2,b))}$$

$$h^a_{b_1,b_2} = d_{((a,b_1),(a,b_2))}$$

be metrics for graphs G and H.

Observe that $\forall a_1, a_2, b_1, b_2,\ d_{((a_1,b_1),(a_2,b_2))} \leq g^{b_1}_{a_1,a_2} + h^{a_2}_{b_1,b_2}$. This is because d is a metric, and so any path between (a_1, b_1) and (a_2, b_2) must be of weight $\geq d_{((a_1,b_1),(a_2,b_2))}$. Therefore in particular, so is the path from (a_1, b_1) to (a_2, b_2) that goes through (a_2, b_1). But the length of this path is exactly $g^{b_1}_{a_1,a_2} + h^{a_2}_{b_1,b_2}$. Hence the observation.

Since $L(G \times H)$ is achieved by d, we have that

$$\begin{aligned}
L(G \times H) &= \frac{2\sum_{((a_1,b_1),(a_2,b_2))\in E} d_{(a_1,b_1),(a_2,b_2)}}{\sum_{(a_1,b_1)\neq(a_2,b_2)} d_{(a_1,b_1),(a_2,b_2)}} && (1)\\
&= \frac{2(\sum_b \sum_{(a_1,a_2)\in E_G} g^b_{(a_1,a_2)} + \sum_a \sum_{(b_1,b_2)\in E_H} h^b_{(b_1,b_2)})}{\sum_{(a_1,b_1)\neq(a_2,b_2)} d_{(a_1,b_1),(a_2,b_2)}} && (2)\\
&\geq \frac{2\sum_b \sum_{(a_1,a_2)\in E_G} g^b_{(a_1,a_2)} + 2\sum_a \sum_{(b_1,b_2)\in E_H} h^b_{(b_1,b_2)}}{\sum_{(b_1\neq b_2)} \sum (a_1 \neq a_2)(g^{b_1}_{(a_1,a_2)} + h^{a_2}_{(b_1,b_2)})} && (3)\\
&= \frac{2\sum_{(a_1,a_2)\in E_G} \sum_b g^b_{(a_1,a_2)} + 2\sum_{(b_1,b_2)\in E_H} \sum_a h^b_{(b_1,b_2)}}{\sum_{(a_1,a_2)} \sum_b g^b_{(a_1,a_2)} + \sum_{(b_1,b_2)} \sum_a h^a_{(b_1,b_2)}} && (4)\\
&\geq \min(\frac{2\sum_{(a_1,a_2)\in E_G} \sum_b g^b_{(a_1,a_2)}}{\sum_{(a_1,a_2)} \sum_b g^b_{(a_1,a_2)}}, \frac{2\sum_{(b_1,b_2)\in E_H} \sum_a h^a_{(b_1,b_2)}}{\sum_{(b_1,b_2)} \sum_a h^a_{(b_1,b_2)}}) && (5)\\
&\geq \min(\frac{\alpha}{V_H}, \frac{\beta}{V_G}). && (6)
\end{aligned}$$

(1) follows from the definition of $L(G \times H)$, (2) follows from the definition of g^b and h^a, (3) follows from the observation, (4) follows from rearrangement of terms, (5) follows from the fact that $\frac{A_1+B_1}{A_2+B_2} \geq \min(\frac{A_1}{A_2}, \frac{B_1}{B_2})$, (6) follows from the definition of $L(G)$ and $L(H)$.

We have shown that

$$L(G \times H) = \min(\frac{\alpha}{|V_H|}, \frac{\beta}{|V_G|})$$

Now the last thing that remains to be shown is why d must assign weight 0 to all edges of the form $((a, b_1), (a, b_2))$ if $\frac{\alpha}{|V_H|} < \frac{\beta}{|V_G|}$. It follows from the observation that if $\frac{A_1+B_1}{A_2+B_2} = \frac{A_1}{A_2}$, and $\frac{B_1}{B_2} < \frac{A_1}{A_2}$, then we know that $B_1 = B_2 = 0$. Since this is the case now, we know that $\sum_{(b_1,b_2)} \sum_a h^a_{(b_1,b_2)} = 0$. Since a metric cannot take negative values, in order for the above to be true, $\forall a, b_1, b_2,\ h^a_{(b_1,b_2)} = 0$. By definition of h^a, this implies that $d_{((a,b_1),(a,b_2))} = 0$. ■

Now we need to use the lemma to show that the cut that the linear programming relaxation suggests for the product graph of an expander and a path, is far from the optimal one by a factor of $O(\frac{1}{\log n})$. Let G be an expander on n nodes, c is a parameter and H be a path graph of length $\frac{\log n}{c}$. Then, by the previous examples, $L(G) = \alpha = \frac{1}{O(n \log n)}$ and $L(H) = \beta = \frac{4c^2}{\log^2 n}$.

Observe that for sufficiently small values of c and sufficiently large values of n, $\frac{\alpha}{|V_H|} > \frac{\beta}{|V_G|}$. That is because $\frac{\alpha}{|V_H|} = \frac{\alpha c}{\log n} = \frac{c}{O(n \log^2 n)}$ and $\frac{\beta}{|V_G|} = \frac{\beta}{n} = \frac{4c^2}{n \log^2 n}$.

If $\frac{\alpha}{|V_H|} > \frac{\beta}{|V_G|}$, then the metric suggested by the linear programming relaxation will assign 0's to all the edges from the path graph, and so the cut won't cut a single edge that came from the path graph, but will instead cut at least $O(n)$ edges per each expander graph, for a total of $O(n \log n)$ edges. At the same time, the optimal cut only cuts n edges. Therefore, the cut suggested by the linear programming approximation is $O(\log n)$ far from optimal.