18.409 Applied Extremal Combinatorics	**Fall 1998**

Lecture 9 : October 15, 1998

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1 Random Expanders

In the previous lecture, we assumed the existence of expanders. In this section, we will show that expanders actually exist. Perhaps the easiest way to do this is via the probabilistic method, showing that a random d-regular graph is likely to be an expander.

For completeness, let us restate the definition of an expander graph.

Definition 1 *Let d be a fixed integer. A family $\mathcal{F}$ of d-regular graphs is said to be a family of expander graphs if there exists a constant ϵ such that every graph $G = (V, E)$ in $\mathcal{F}$ satisfies the following expansion property:*

$$\forall S \subset V \ \ s.t. \ |S| \leq \frac{n}{2} \qquad |N(S)| \geq (1+\epsilon)|S| \tag{1}$$

where $N(S) = \{x : \exists y \in S such\ that (x, y) \in G\}$.

Although it is possible to show that a random d-regular graph is a good expander, we will show only that a random d-regular bipartite graph is a good expander. Our method of choosing a random d-regular bipartite graph will be to choose d random (perfect) matchings between two sets of n nodes and take their union. We will probably end up with some nodes whose degree is less than d, but we can add edges to make it d-regular. We will show that the original graph (before adding the extra edges) is a good expander with high probability; adding edges certainly won't change this. Specifically, we will prove that all sets of αn nodes on the left side have more than βn neighbors on the right, for constants $\alpha, \beta < 1$ and sufficiently large n. Let us call the set of left-nodes U and right-nodes V. Consider any pair of sets $A \subset U$ and $B \subset V$ with $|A| = \alpha n$ and $|B| = \beta n$. Observe that for a single random perfect matching, the probability that all the neighbors of A are in B is $\frac{\binom{\beta n}{\alpha n}}{\binom{n}{\alpha n}}$. Since we have chosen our d perfect matchings independently,

$$\Pr[N(A) \subset B] = \left(\frac{\binom{\beta n}{\alpha n}}{\binom{n}{\alpha n}} \right)^d \leq (\beta^{\alpha n})^d$$

Therefore, for a random $A \subset U$ with $|A| = \alpha n$, we have

$$\Pr[|N(A)| \leq \beta n] \leq \sum_{|B|=\beta n} \Pr[N(A) \subset B] \leq \binom{n}{\beta n} \beta^{d\alpha n}$$

and hence

$$\Pr[\exists A \subset U \text{ s.t. } |A| = \alpha n, |N(A)| \leq \beta n] \leq \sum_{|A|=\alpha n} \Pr[|N(A)| \leq \beta n] \leq \binom{n}{\alpha n}\binom{n}{\beta n} \beta^{d\alpha n} \quad (2)$$

We can now apply the estimate $\binom{n}{\gamma n} \sim 2^{nH(\gamma)}$, where $H(\gamma) = -\gamma \log_2 \gamma - (1-\gamma)\log_2(1-\gamma)$ is the binary entropy function. Thus, the RHS of (2) is asymptotic to

$$2^{(H(\alpha)+H(\beta)+d\alpha \log_2 \beta)n}. \quad (3)$$

The idea now is to pick α and β such that $H(\alpha) + H(\beta) + d\alpha \log_2 \beta = -r$ is negative. If we do that, then our probability is bounded above by something asymptotic to 2^{-rn}, which goes to zero as n goes to infinity. Then we will be able to conclude that, if we choose at random a d-regular bipartite graph on $2n$ vertices (with n large), then it is extremely unlikely to find a set of αn left-nodes which expands to a set of fewer than βn right-nodes.

To make this more concrete, set $d = m + \log_2 m + 1$, $\alpha = 1/m$, and $\beta = 1/2$. Then it can be checked that $H(\alpha) + H(\beta) + d\alpha \log_2 \beta < 0$, so (3) tends to 0 as $n \to \infty$. This proves that, in a random d-regular bipartite graph (where $d = m + \log_2 m + 1$), all sets of n/m vertices expand to at least $n/2$ vertices, for an expansion factor of about $d/2$.

This argument can be made to show that all sets of vertices expand sufficiently to meet the criterion (1). In addition, a similar argument can be used to show that a random d-regular graph (not necessarily bipartite) is a good expander.

2 Expanders and Girth

Consider a graph G with very high expansion. If we zoom into a tiny patch of G, we might expect it to look like a tree. Indeed, our intuition tells us that small cycles in a graph tend to decrease its expansion. To make this more precise, we make the following definition.

Definition 2 *The girth of a graph G is the length of the shortest cycle of G.*

Consider any vertex v in a graph G with girth g. If we consider all the walks of length less than or equal to $\frac{g-1}{2}$ which start at v and don't back up on themselves, then all the endpoints of these walks are distinct. This means that v, the neighbors of v, the neighbors of their neighbors, and so forth, are all distinct. For that reason, we can expect that graphs with a large girth will make good expanders.

This leads us to consider the problem of determining how large the girth of a graph can be. We can get a straightforward upper bound on the girth of a graph in terms of its minimum vertex degree d. Since the number of walks of length $\frac{g-1}{2}$ starting at v is less than n, we have the inequality $n > d^{\frac{g-1}{2}}$, which we can rewrite as $g < 1 + 2\log_d n$.

In the other direction, the following theorem tells us that it is possible to obtain $g = \log_{2d} n$.

Theorem 3 (Erdos-Sachs) *There exists a graph on at most $(2d)^g$ vertices with girth g in which every vertex has degree at least d.*

Proof: We use a probabilistic argument.

Let $n = (2d)^g$. First we claim that there exists a graph on at most n vertices and dn edges which has at most $n-1$ cycles of length less than g. To show this, pick a random graph G on n vertices by choosing $M = dn$ edges at random. Consider the random variable Z_l which counts the number of cycles of length l in G. We can compute the expected value of Z_l. Since there are $\frac{1}{2}\binom{n}{l}(l-1)!$ possible cycles of length l in G, and each one of them occurs with probability $\binom{N-l}{M-l}/\binom{N}{M}$ (where $N = \binom{n}{2}$) ,we have

$$E[Z_l] = \frac{1}{2}(l-1)!\binom{n}{l}\frac{\binom{N-l}{M-l}}{\binom{N}{M}}.$$

Applying the bounds

$$\frac{\binom{N-l}{M-l}}{\binom{N}{M}} \leq \left(\frac{M}{N}\right)^l = \left(\frac{2d}{n-1}\right)^l, \qquad (l-1)!\binom{n}{l} \leq \frac{n(n-1)^{l-1}}{l}$$

we get

$$E[Z_l] \leq (2d)^l.$$

Therefore the expected number of cycles of length less than g in G is

$$\sum_{l=1}^{g-1} E[Z_l] \leq \sum_{l=1}^{g-1}(2d)^l < (2d)^g = n.$$

We conclude that, for at least one such graph G_0, the number of cycles of length less than g is at most $n-1$.

We can break each one of these cycles by deleting one of its edges. If we do that, we get a graph G_1 of girth g with n vertices and at least $dn - (n-1)$ edges. Now we just need to find a non-empty subgraph H of G_1 such that every vertex has degree at least d. This H will clearly satisfy the conditions of the problem.

To do this, we delete, one by one, the vertices of G_1 with degree less than d. More precisely, if the graph G_k has vertices of degree less than d, then we let G_{k+1} be the graph obtained by removing one such vertex and its incident edges. We repeat this process until some G_K has no vertices of degree less than d. Then we just need to show that $G_K = H$ is a non-empty graph, and we will be done.

Notice that, since G_1 has n vertices, and we remove one vertex in each step, $K \leq n+1$. But then, since G_1 has $(d-1)n+1$ edges, and we remove at most $d-1$ edges in each step, G_K has at least $(d-1)n+1-(d-1)(K-1) = (d-1)(n+1-K)+1 \geq 1$ edges. It follows that G_K is non-empty, and the proof of the theorem is complete.

3 Expanders and the Spectrum

Although the probabilistic argument of the previous section is a simple way to prove that expanders exist, we would prefer to have explicit constructions of expanders. We will analyze such constructions by using λ_2, the second largest eigenvalue of the adjacency matrix. Recall Tanner's theorem, which we proved in lecture 2.

Theorem 4 (Tanner) *Let M be the adjacency matrix of a d-regular graph G and let λ_2 be its second largest eigenvalue. Then, for all sets A of size αn,*

$$N(A) \geq \left(\frac{d^2}{d^2\alpha + \lambda_2^2(1-\alpha)} \right) \alpha n$$

(Note: for small α, this gives expansion like d^2/λ_2^2.)

Tanner's theorem reduces the problem of finding good expanders to finding graphs for which the second-largest eigenvalue is large. A result which we won't prove, attributed to A. Nilli, but written with the help of N. Alon, is the following.

Theorem 5 *No infinite familiy of d-regular graphs has the second largest eighenvalue λ_2 bounded below $2\sqrt{d-1}$.*

This tells us that, using Tanner's theorem, we can only hope to guarantee expansion of about $\frac{d}{4}$ for small sets. In fact, using Cayley graphs, Margulis, and Lubotzky, Phillips and Sarnak gave a construction of graphs with $\lambda_2 \leq 2\sqrt{d-1}$, thus achieving that expansion factor. However, we already know that random graphs do better than this, achieving an expansion factor of almost $d-1$ for sufficiently small sets. Kahale was able to improve the estimate in Tanner's theorem to show that when $\lambda_2 = 2\sqrt{d-1}$, we actually get expansion of about $\frac{d}{2}$ for small sets. He also showed that there exist d-regular graphs with $\lambda_2 \leq 2\sqrt{d-1}$ in which there exist two vertices with the same set of neighbors; so, the expansion of the graph is at best $\frac{d}{2}$.