Advanced Complexity Theory
18.405/6.841
Due: Thursday, March 15th, 2001

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Problem Set 1

1 Show that if PSPACE = PH $\left(=\bigcup_k \Sigma_k \mathrm{P}\right)$ then PH has only finitely many distinct levels.

2 Show that for any circuit C comprised of AND, OR and NOT gates, there is a circuit C' comprised of AND, OR, and NOT gates that computes the same function as C such that

a. the only NOT gates in C' are directly above the inputs, and

b. the size of C' is at most twice the size of C. In this case, we measure the size of a circuit by the number of AND and OR gates it has.

3 P/log is the class of languages L such that there exists a polynomial-time turing machine M, a constant c, and a sequence of strings $\{A_i\}_{i\in\mathbf{N}}$, such that $|A_i| < c\log i$, and

$$w \in L \text{ if and only if } M \text{ accepts input } (w, A_{|w|})$$

Show that if $NP \subseteq P/log$, then $P = NP$.

4 A function $f : \{0,1\}^n \mapsto \{0,1\}$ is said to have *hardness* $h(n)$ if for all circuits C with at most $h(n)$ gates,

$$\left| \operatorname*{Prob}_{x\in\{0,1\}^n} [C(x) = f(x)] - 1/2 \right| < 1/h(n).$$

Prove that for sufficiently large n there exist functions of hardness $2^{n/8}$. (Note: this is far from tight)

5 Show that if $\Sigma_k^p/poly = \Pi_k^p/poly$, then $\Sigma_{k+2}^p = \Pi_{k+2}^p$.

6 Prove that $\mathrm{P}^A = \mathrm{BPP}^A$ for a random oracle A, with probability 1. (*i.e.,* show that it holds with probability greater than $1-\epsilon$, for all $\epsilon > 0$.) By a random oracle, I mean that each word is in the oracle with probability 1/2, and each word is independent of every other word. If there is a fact from probability that you need, but have not learned, then just state it and assume it. (Warning: a BPP^A machine could do strange things, like use its input x to index a string in the oracle A and then check if that string is in a BPP language.)

Homework policy:

If you work with anyone else on the homework, please give them due credit (i.e., list who you worked with on which problems). Cite any sources you use, but please don't look up the answer. If you don't know the answer to a problem, then just don't answer it. Do not write anything you don't believe. Avoid making yourself believe a false proof—it damages your brain.