18.409 The Behavior of Algorithms in Practice 4/2/2002

Lecture 13

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In today's class, we will:

- Analyze von Neumann's algorithm in terms of the condition number [this work is due to Dantzig and Eppelman-Freund]
- Do a smoothed analysis of the condition number

1 von Neumann's Linear Programming Algorithm

We consider the following formulation of linear programming:
Given $c, a_1, a_2, \ldots, a_n \in \Re^d$
Is $c \in hull\{a_1, a_2, \ldots, a_n\}$?
In other words, does there exist $x_1, x_2, \ldots, x_n$ such that:

$$\sum a_i x_i = c$$

$$\sum x_i = 1$$

$$x_i \geq 0$$

We would like to find either an x s.t $\|c - \sum a_i x_i\| < \epsilon$ if the linear program is feasible or find a hyperplane separating c from $a_1, a_2, \ldots, a_n$ if it is not.
For technical reasons, we'll assume that the origin lies within the hull.

Recall von Neumann's algorithm for this program.

1. Let $y = Ax$
2. Choose i maximizing $\langle c - y, a_i - y \rangle$
3. If $\langle c - y, a_i - y \rangle / \|c - y\| < \|c - y\|$ then return infeasible
4. Otherwise set x' s.t. $y' = Ax'$ is the point on the line $\overline{ya_i}$ closest to c. Repeat.

The correctness of the algorithm derives from the following:
Claim: If $y, c \in hull\{a_1, a_2, \ldots, a_n\}$ then $\exists i$ s.t.

$$\langle \frac{c-y}{\|c-y\|}, a_i - y \rangle \geq \|c-y\|$$

2 Analysis of the algorithm

Let r =dist(c,bdry(ch$\{a_1, a_2, \ldots, a_n\}$))

Let $R = \max_i \|a_i\|$

Theorem 1 (Eppelman-Freund)
(I) If the linear program is feasible, the algorithm finds x s.t. $\|Ax - c\| \leq \epsilon$ in at most $8(R/r)^2 ln(2R/\epsilon)$ iterations.
(II) If infeasible, the algorithm discovers this in at most $4(R/r)^2$ iterations.

Proof
Part (I)
Claim: If the linear program is feasible then

$$\|Ax' - c\| \leq \|Ax - c\|\sqrt{1 - (\frac{r}{2R})^2}$$

Before proving this claim, let's see why it implies (I). Observe that

$$\sqrt{1 - (\frac{r}{2R})^2} \leq \sqrt{e^{-(r/2R)^2}}$$

Initially,

$$\|Ax_0 - c\| \leq 2R$$

Therefore after k iterations,

$$\|Ax_k - c\| \leq 2R\sqrt{e^{-k(r/2R)^2}}$$

If we let $2R\sqrt{e^{-k(r/2R)^2}} < \epsilon$ we get (I).

Proof of claim

¿From Fig 1. it is clear that,

$$\text{dist}(c, y') \leq \text{dist}(c, y^{\perp})$$

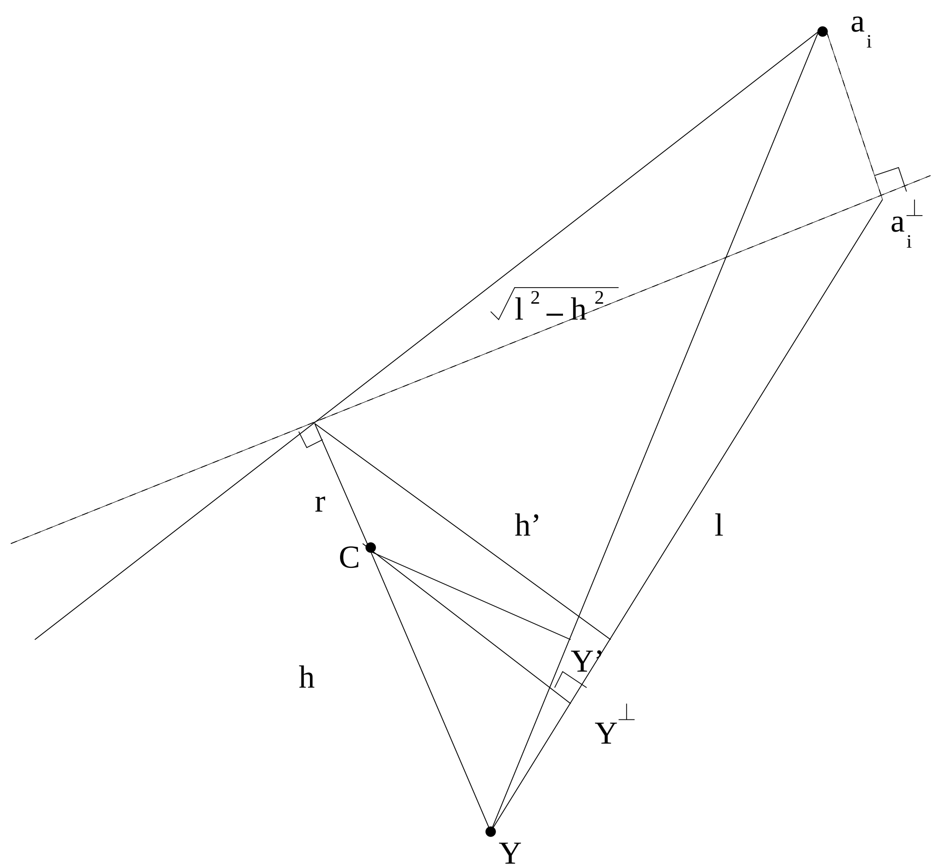

Figure 1: Getting a bound on $dist(c, y')$

By similarity of triangles,

$$\frac{\text{dist}(c, y^{\perp})}{\text{dist}(c, y)} = \frac{h'}{h}$$

Therefore,

$$\begin{aligned}\frac{dist(c, y')}{dist(c, y)} &\leq \frac{h'}{h} \\ &= \frac{\sqrt{l^2 - h^2}}{l} \\ &= \sqrt{1 - (h/l)^2}\end{aligned}$$

Since, $h \geq r$ and $l \leq 2R$, it follows that

$$\frac{dist(c, y')}{dist(c, y)} \leq \sqrt{1 - (\frac{r}{2R})^2}$$

Part (II)

Claim (Dantzig) After k iterations,

$$\|Ax - c\| \le \frac{2R}{\sqrt{k}}$$

Again it's not hard to show why this implies (II). Since point y cannot go outside the convex hull, the algorithm will stop once

$$\frac{2R}{\sqrt{k}} \le r$$

So the algorithm must terminate after $(2R/r)^2$ iterations.

Proof of claim

Let h_k be the height (i.e., distance between c and y) after k^{th} iteration. As in the previous case, we can show that,

$$h_{k+1} \le h_k \sqrt{1 - (h_k/2R)^2}$$

We claim that

$$h_k \le 2R/\sqrt{k}$$

Proof by induction:

Note that $\|c\| \le R$ since otherwise it would be trivial to decide infeasibility.

Therefore, $h_1 \le 2R$

Let $h_k \le 2R/\sqrt{k}$

Then

$$\begin{aligned} h_{k+1} &\le h_k \sqrt{1 - (h_k/2R)^2} \\ &= 2R/\sqrt{k}\sqrt{1 - 1/k} \\ &= 2R\sqrt{\frac{k-1}{k^2}} \\ &\le 2R/\sqrt{k+1} \end{aligned}$$

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3 Condition number of a linear program

Given a linear program, $c, a_1, a_2, \ldots, a_n$, we define it's condition number, K, as follows:

If $c \in hull\{a_1, a_2, \ldots, a_n\}$,
$K(c, a_1, a_2, \ldots, a_n) = inf\{\|\Delta c\| + \sum_i \|a_i\| : c + \Delta c \notin hull\{a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_n + \Delta a_n\}\}$

In other words, K measures the smallest change to infeasibility. Similarly,

If $c \notin hull\{a_1, a_2, \ldots, a_n\}$,
$K(c, a_1, a_2, \ldots, a_n) = inf\{\|\Delta c\| + \sum_i \|a_i\| : c + \Delta c \in hull\{a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_n + \Delta a_n\}\}$

Claim 2 $K(c, a_1, a_2, \ldots, a_n) = r$

It is obvious that $K(c, a_1, a_2, \ldots, a_n) \leq r$

We'll try to prove $K(c, a_1, a_2, \ldots, a_n) \geq r$

Lemma 3 *Let* $C = hull\{a_1, a_2, \ldots, a_n\}, C' = hull\{a_1 + \Delta a_1, a_2 + \Delta a_2, \ldots, a_n + \Delta a_n\}$

1. $\gamma(C, C') \leq \max_i \|\Delta a_i\|$
2. $\gamma(\bar{C}, \bar{C}') \leq \max_i \|\Delta a_i\|$
3. $\gamma(bdry(C), bdry(C')) \leq \max_i \|\Delta a_i\|$

where $\gamma(A, B) = \max_{x \in A} \min_{y \in B} dist(x, y)$

Proof

1. Let $x \in C$
   $\Rightarrow \exists \alpha_1, \alpha_2, \ldots, \alpha_n$ s.t.

$$\begin{aligned} \sum \alpha_i &= 1 \\ \alpha_i &\geq 0 \\ x &= \sum_i \alpha_i a_i \end{aligned}$$

   Let $x' = \sum \alpha_i (a_i + \Delta a_i) \in C'$. Then,

$$\begin{aligned} dist(x, x') &= \|\sum \alpha_i \Delta a_i\| \\ &\leq (\sum \alpha_i) \max_i \|\Delta a_i\| \\ &= \max_i \|\Delta a_i\| \end{aligned}$$

2. This is similar to 1.

3. Let $x \in bdry(C)$
   $\exists x_1 \in C' \, s.t. \, dist(x, x_1) \leq \max_i \|\Delta a_i\|$
   Similarly, $\exists x_2 \in \bar{C}' \, s.t. \, dist(x, x_2) \leq \max_i \|\Delta a_i\|$
   Therefore, on the line from x_1 to x_2 $\exists$ a point on $bdry(C')$ with distance from x at most $\max_i \|\Delta a_i\|$

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It takes a little more work to actually use this lemma to prove the fact that $K \geq r$. Intuitively, what it means is that the boundary of the convex hull doesn't move much by changing the a_i's. Therefore, it is better to just change c.

In the next lecture, John Dunagan will do a smoothed analysis of the condition number. In particular, we shall prove the following theorem.

Theorem 4 *For $c, a_1, a_2, \ldots, a_n$ Gaussian random vectors with variance σ^2 and centered at $\bar{c}, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$, such that each has norm ≤ 1,*

$$Pr[K(c, a_1, a_2, \ldots, a_n) < \epsilon] \leq \frac{128 d^{1/2} \epsilon}{\sigma}$$

This result actually follows trivially from an earlier result, but with a less intuitive proof.

Theorem 5 (Keith Ball '93) *For any convex body K, and Gaussian random vector c,*

$$Pr[dist(c, bdry(K)) < \epsilon] \leq \frac{8 d^{1/4} \epsilon}{\sigma}$$