18.409 The Behavior of Algorithms in Practice	4/30/2002
Lecture 18	
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1 left off:

$Prob[ang(z, \delta\Delta(a_{\pi_1},...,a_{\pi_d})) < \epsilon \mid opt\Delta_2(a...a_n) = \pi_1...\pi_d]$ where $a...a_n$ are the dist. according to $\prod_{i=1}^{n} \mu_i(\alpha_i)$, μ_i are Gaussian, $var\sigma^2 \leq 1$, center norm ≤ 1.

2 key idea:

change of variables

$$a_{\pi_1} \dots a_{\pi_d} \rightarrow \parallel w \parallel = 1, r \geq 0 \text{ such that } < w, a_{\pi_i} >= r$$

$$b_{\pi_1} \dots b_{\pi_d}$$

be local coords. of $a_{\pi_1} \dots a_{\pi_d}$ on that plane.

observe that unlikely $\parallel a_i \parallel \geq 1 + 4\sqrt{d \log n}$ so let $\Gamma = 1 + \sqrt{d \log n}$.

So can say:

$$ang(z, \delta\Delta(a_{\pi_1}, \dots, a_{\pi_d})) \geq \frac{dist(z^{w,r}, \delta\Delta(b_{\pi_1}, \dots, b_{\pi_d})) < z, w >}{3r}$$

Let $z^{w,r}$ be a point along ray direction on plane w,r, need to bound:

$$Pr_{w,r,b_{p_1},\dots,b_{p_d}}[\frac{dist(z, \delta\Delta(b_{\pi_1}, \dots, b_{\pi_d})) < z, w >}{3r} < \epsilon]$$

where $w, r, b_{p_1}, \dots, b_{p_d}$ have density:

$$[\prod_{j \in \pi_1 \dots \pi_d} \int_{a_j} [< w, a_j > \leq r] \mu_i(a_j)](\prod_{j=1}^{d} \mu_{\pi_i}(a_{\pi_i}))[z^{w,r} \in \Delta(b_{\pi_1}, \dots, b_{\pi_d})] Vol(\Delta(b_{\pi_1}, \dots, b_{\pi_d}))$$

(suitably re-normalized)

for today show: $\forall z \forall w, r$ such that $\parallel z^{w,r} \parallel \leq \Gamma$

$$Pr_{b_1}, \dots, b_d[dist(z^{w,r}, \delta\Delta(b_{\pi_1}, \dots, b_{\pi_d})) < \epsilon] < \frac{16 e^4 \Gamma^2 (1 + \Gamma)}{\sigma^4} \epsilon$$

Where $b_{\pi_1},\ldots,b_{\pi_d}$ have density $(\prod_{i=1}^{d}\mu\pi_i(a_{\pi_i}))Vol(\Delta(b_{\pi_1},\ldots,b_{\pi_d}))[z^{w,r}\in\Delta(b_{\pi_1},\ldots,b_{\pi_d})]$ where the first factor of the very last formula is in terms of b_{π_i}s Gaussian each.
$Var\sigma^2$ center norm ≤ 1.

Theorem 1. *re-stated*

Let $b_1\ldots b_d$ be points in $\Re^{d-1}$

with density $\prod_{i=1}^{d}\mu_i(b_i)Vol(\Delta(b_1,\ldots,b_d))[z\in\Delta(b_1,\ldots,b_d)]$ where $\parallel z\parallel\leq\Gamma,\mu_i$ Gaussian, $var\sigma^2\leq 1$, norm center ≤ 1.

What is $Pr[dist(z,\delta\Delta(b_1,\ldots,b_d))<\epsilon]$?

Two steps:

1. Show b_1 unlikely near aff. $(b_1\ldots b_d)$

2. Given b_1 is far from aff. $(b_1\ldots b_d)$, unlikely too close to aff. $(b_1\ldots b_d)$. (where aff. denotes an affine line)

to 1: $Pr[dist(b_1,aff.(b_2\ldots b_d))<\epsilon]<e^2(\frac{d(1+\Gamma)\epsilon}{\sigma^2})^3$

to 2: $Pr[dist(z,aff.(b_2\ldots b_d))<\epsilon dist(b_1,aff.(b_2\ldots b_d))<\frac{2e^3d(2p)^2\epsilon}{\sigma^2}]$

These imply Theorem 1. We use change of variables to show 1:

Make z the origin (for notational simplicity), so now the distances have centers of norm $\leq\Gamma+1$.

Let $\parallel\tau\parallel=1,t\geq 0$ be such that $<b_i,\tau>=t$ for $i=1..d$ and let $c_2..c_d$ be the local coords. of the b_is in that plane. Jac is $Vol(\Delta(c_2..c_d))$

$$dist(b,aff.(b_2..b_d))=t-<b_1,\tau>$$

Let $l=-<b_1,\tau>$, let c_1 be the projection of b_1 onto the plane spee by τ,t.

$Pr_{\tau,t,l,c_1,\ldots,c_d}[(l+t)<\epsilon]$ with density

$$(\prod_{i=1}^{d}\mu_i(b_i))Vol(\Delta(b_1\ldots b_d))[O\in\Delta(b_1\ldots b_d)]Vol(\Delta(c_2\ldots c_d))$$

where $Vol(\Delta(c_2\ldots c_d))$ equals $(l+t)Vol(\Delta(c_2\ldots c_d))\sim(1+\alpha)t$ (*)

Note: for any $c_1,c_2,\ldots,c_d,l,t$ such that $O\in\Delta(b_1,\ldots,b_d)$ if mult. l,t by const.

$\rightarrow$ cl ct still have $O\in\Delta(b_1,\ldots,b_d)$.

let $l=\alpha t$ gives Jacobian $|\frac{\delta l}{\delta\alpha}|=t$

$(*)\sum\max_{c_1,\ldots,c_d,t,\alpha,s.t.O\in\Delta(b_1,\ldots,b_d)}Pr[(1+\alpha)t<\epsilon]$

where t has density

$(\prod_{i=1}^{d}\mu_i(b_i))t^2$(, near O in small region looks like a constant)

We will actually bound

$$Pr[tmax(1,\alpha) < \epsilon] \geq Pr[(1+\alpha)t < \epsilon]$$

$b_2 = (t, c_2)$

$b_1(-\alpha t, c_1)$

can write this way:

μ is centered at points of norm $\leq 1 + \Gamma$

So,we claim for $0 \leq t \leq \frac{\sigma^2}{d(1+\Gamma)}$

$\mu_i(t, b_i)$ varies by at most $e^{\frac{2}{d}}$ (former, we showed similar with no ds)

For $0 \leq t \leq \frac{\sigma^2}{d(1+\Gamma)}$

$b\mu_1(-\alpha t, b_1)$ varies by at most $e^{\frac{2}{d}}$, setting $t_0 = \frac{\sigma^2}{d(1+\Gamma)} min(1, \frac{1}{\alpha})$ then

$$\frac{\max_{0\leq t\leq t_0} \prod_{i=1}^{d} \mu_i(b_i)}{\min_{0\leq t\leq t_0} \prod_{i=1}^{d} \mu_i(b_i)} \leq \epsilon^2 \ (**)$$

So by the following Proposition 1, we have

$Pr[t < \epsilon t_0] \leq e^2\epsilon^3$ or $Pr[tmax(1,\alpha) < \frac{\epsilon\sigma^2}{d(1+\Gamma)}] \leq e^2\epsilon^3$

$\Rightarrow Pr[tmax(1,\alpha) < \epsilon] \leq e^2(\frac{d(1+\Gamma)}{\sigma^2}\epsilon)^3$

Proposition 1. *Let t be distributive according to $f(t)t^2$ where*

$$\frac{\max_{0\leq t\leq t_0} f(t)}{\min_{0\leq t\leq t_0} f(t)} \leq \epsilon^2$$

then

$$Pr[t < \epsilon t_0] \leq e^2\epsilon^3$$

Pf 1. $Pr[t < \epsilon t_0] \leq \frac{Pr[t<\epsilon t_0]}{Pr[t<t_0]} = \frac{\int_{t=0}^{\epsilon t_0} f(t)t^2}{\int_{t=0}^{t_0} f(t)t^2} \leq \frac{\max_{0\leq t\leq \epsilon t_0} f(t)}{\max_{0\leq t\leq t_0} f(t)} \frac{\int_{t=0}^{\epsilon t_0} t^2}{\int_{t=0}^{t_0} t^2} \leq e^2\epsilon^3$