18.409 The Behavior of Algorithms in Practice	3/12/2002
Lecture 9	
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1 Well shaped mesh

Split a partition space into simplices and get a graph on vertices:

$$\text{aspect ration}(\Delta) = \frac{\text{largest edge}}{\min_{vertices} \text{distance vertices to plane through opponent face}} \geq 1$$

$$well - shaped = \text{bounded aspect ratio}$$

History:

Miller - Thorsten

Teng - Vavasis (K-nearest neighbour graph)

Every k-nearest neighbour graph in $\Re^d$ is a $k\tau_\alpha$ - ply intersection graph

Definition 1. *A k-ply intersection graph comes from a set of balls B_1,...,$B_n \in \Re^d$ such that no point lies in the interior of more than k balls.*

(i,j) $\in E$ if $B_i \cap B_j \neq \emptyset$

τ_δ = kissing number in $\Re^d$

(max. number of unit balls through another unit ball: $\tau_2 = 6$)

(planar is a 1-ply intersection graph)

An α-overlap graph is a set of interior disjoint balls $B_1, ...B_n$:

(i,j) $\in E$ if $\alpha B_i \cap B_j \neq \emptyset$ and $B_i \cap \alpha B_j \neq \emptyset$

Theorem 1. *MTTV*

Every well shaped mesh is a bounded degree subgraph of an α-overlap graph.

Pf 1. -

Claim 1. *for aspect ratio $\leq \beta$,then graph has degree $\leq f(\beta)$. Can lower bound angle of a corner of a simplex, so can upper bound number of simplices at a vertex.*

For every vertex, locate a ball at that vertex of radius 0.5 the shortest edge leaving that vertex.

Need to show:

$\frac{\textit{longest edge on vertex}}{\textit{shortest edge on vertex}}$ *is bounded* $\leq \beta^{\textit{number of neighbours of vertex}}$

Look at a band matrix with bandwith b:

For an ordering $\tau : V \to [1..n]$ bandwith of G under τ

$$\Phi(G,\tau) = \min_b \mid \tau(u) - \tau(v) \mid > b$$

$$\Rightarrow (u,v) \notin E$$

Bandwith of G is $\min_\tau \Phi(G,\tau) = \Phi(G)$

Cuthill - Mc Kee used BFS: outputs τ such that:

$$d(\tau^{-1}(1)) < d(\tau^{-1}(1), u)$$

$$\Rightarrow \tau(u) < \tau(v)$$

$G(n,p)$ graph on n noodes in which each edge (i,j) is chosen to be in graph with prob p indepently.

Theorem 2. *For almost all* $G \leftarrow G(n,p)$

$$\Phi(G) \geq n - 4log_{\frac{1}{1-p}} n$$

Pf 2. *Claim follows if* $\Phi(G) \geq n - 2k$*:* $\exists U_1, U-2 : \mid U_1 \mid = \mid U_2 \mid = U$ *no edges between* U_1 *and* U_2*. Set* $k = 2\log_{\frac{1}{1-p}} n$:

$$\binom{n}{k}\binom{n-k}{k}(1-p)^{k^2} \leq (\frac{e^n}{n})^{2^k}(\frac{1}{n^2})^k = \binom{e}{k}^2 k \to 0$$

Turner: define $G_b(n,p)$ same as $G(n,p)$, but not edges for $\mid i-j \mid > b$ (Bandmatrix)

1) $\Phi(G_b(n,p)) \geq b - 4\log_{\frac{1}{1-p}} b$ almost always

2) A level-set heuristic returns τ at $\Phi(G,\tau) \leq 3(1+\epsilon)b \; \forall\epsilon > 0$ for almost all $G \leftarrow G_b(n,p)$

to proof 1) we use the same arguments as before, 2): Let $V_i = u : dist(u,1) = i$

Theorem 3. *for almost all* $AA : G \leftarrow G_b(n,p) \; \forall\epsilon > 0, b \geq (1+\epsilon)\log_{\frac{1}{1-p}^2} n$

$$\mid V_1 \mid \leq (1+\epsilon)pb$$

$$\mid V_2 \mid \leq (1+\epsilon)(2-p)b$$

$$i \geq 3 : \mid V_i \mid \leq \frac{1+\epsilon}{b}$$

$$\Phi(G,\tau) \leq \max_i \mid V_i \cup V_{i+1} \mid$$

Lemma 1. *$AA : G \leftarrow G_b(n,p)$ $\forall u, v$ such that $\mid u - v \mid \leq 2b - \alpha$ where $\alpha = (1+\epsilon)\log_{\frac{1}{1-p}^2} n$ exists a path with length ≤ 2 between.*

to prove this show:

The number of possible neighbours of u and r are $2b - \mid u - r \mid \geq \alpha$.

$$\sum_{i=\alpha}^{2b-1} n(1-p^2)^i = n(1-p^2)^\alpha \sum_{i \geq 0} (i - p^2)^i = nn^{-1-\epsilon} - p^{-2} = n^{-2}p^{-2} \to 0$$

Lemma 2. *Let $l_i = \min_i(V_i), r_i = \max_i(V_i)$. $\forall i \geq 3 r_i - l_i \leq b + \alpha$, follows from:*

Lemma 3. $\forall i \geq 3 r_i - 3b \leq r_{i-3} \leq l_i - (2b - \alpha)$