18.413: Error-Correcting Codes Lab	March 11, 2004

Lecture 11

Lecturer: Daniel A. Spielman

11.1 Related Reading

- Fan, Chapter 2, Sections 3 and 4.

11.2 Belief Propagation on Trees

Let's consider the case of low-density parity-check codes when the underlying graph is a tree. These will be useless, but they are how the decoding algorithm is derived. The graph will look like the picture below. Each edge with just one endpoint corresponds to a bit in the code (variable). Each edge with two endpoints corresponds to an artificial internal variable that we introduce for the algorithm. The "=" nodes constrain the variables to which they are attached to be equal (repetition code), and the "+" nodes constrain the variables to which they are attached to have parity 0.

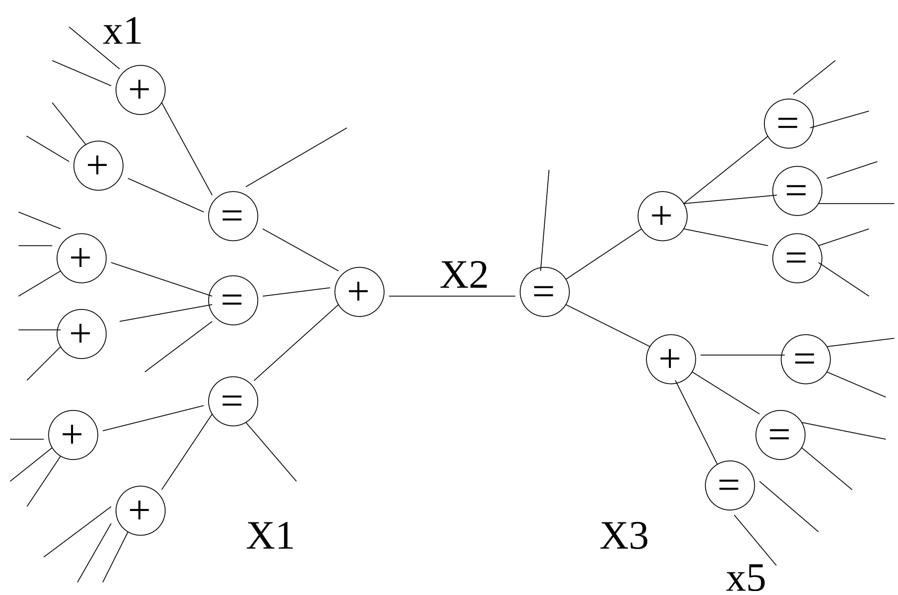


Figure 11.1: A normal form tree: variables live on edges

Let's assume that a random codeword $x_1, \ldots, x_n$ has been transmitted over a channel and that

$b_1, \ldots, b_n$ has been received. We want to compute

$$\mathrm{P}^{post}\left[x_1 = 0 | b_1, \ldots, b_n\right] \sim \mathrm{P}^{ext}\left[x_1 = 0 | b_1\right] \mathrm{P}^{ext}\left[x_1 = 0 | b_2, \ldots, b_n\right]. \quad (11.1)$$

So, we will focus on the computation of $\mathrm{P}^{ext}\left[x_1 = 0 | b_2, \ldots, b_n\right]$. In particular, let's look at what information we need to pass over the edge whose variable in this figure is labeled X_2. To do this, we will group all the variables to the left of X_2 into one big clump, which we call X_1, and those to the right into X_3.

By the three-variable lemma from lecture 6, we have

Lemma 11.2.1.

$$P^{ext}\left[X_1 = a_1 | Y_2 Y_3 = b_2 b_3\right] = \sum_{a_2 : (a_1, a_2) \in \mathcal{C}} P\left[X_2 = a_2 | X_1 = a_1\right] P^{ext}\left[X_2 = a_2 | Y_2 = b_2\right] P^{ext}\left[X_2 = a_2 | Y_3 = b_3\right].$$

.

As X_2 is an internal variable, nothing is received from the channel for it, so there is no Y_2 or b_2. So, the formula simplifies to

$$\mathrm{P}^{ext}\left[X_1 = a_1 | Y_2 Y_3 = b_2 b_3\right] = \sum_{a_2 : (a_1, a_2) \in \mathcal{C}} \mathrm{P}\left[X_2 = a_2 | X_1 = a_1\right] \mathrm{P}^{ext}\left[X_2 = a_2 | Y_3 = b_3\right].$$

. Thus, the only information about X_3 that needs to be passed through variable X_2 is

$$\mathrm{P}^{ext}\left[X_2 = a_2 | Y_3 = b_3\right].$$

Also, since X_2 is a just one bit, we only need this information for $a_2 \in \{0, 1\}$.

Now, I told you that we were interested in x_1, not X_1. But, if we can compute probability estimates for X_1, then we can do the same for x_1. So, the same information is what is required.

This begs the question of how we compute $\mathrm{P}^{ext}\left[X_2 = a_2 | Y_3 = b_3\right]$. The answer is that we look back one step in the tree, and consider the other edges entering the "=" node to the right of X_2. Each of these can pass up a similar message about the subtree beneath them. Then, the "=" node combines all of these estimates according to the rule for the repetition code. Of course, these message came through "+" nodes, which combine their incomming messages according to the rule for parity codes.

11.3 Dynamic programming

This description of computing $\mathrm{P}^{post}\left[x_1 = 0 | b_2, \ldots, b_n\right]$ has one defect: it is doing a whole lot of work just for x_1, and if we want to do it again for some other variable, such as x_5, we would have to do it all over again. However, if you take a look at the algorithm, most of the computations are the same of both variables. In fact, it is possible to perform all the computations for all variables at once while doing little more work that one does for just one variable. The idea is to use the following rule:

> The message a node sends out along an edge should be the corresponding function of the messages that come in on every other edge.

There is a natural was of describing the resulting algorithm in an object-oriented fashion. In this case, each node will correspond to an object. Each node will store all of its incomming messages. As soon as it has an incomming message on all but one of its edges, it sends a message to the node on the other end of the remaining edge. When it has messages on every incomming edge, it sends messages to the nodes at the end of all the other edges. To get this process started, we note that the the channel outputs correspond to a message on each external edge (with one endpoint), so each node that has just one internal edge will be ready as soon as the channel outputs are processed. Also note that, at the end of the algorithm, a message will be sent to each external edge. This will contain the desired extrinsic probability estimation.

The algorithm described above is rather asynchronos: I've described a sufficient algorithm in terms of passing messages, and it works regardless of the order in which messages are delivered (as long as they all get delivered). A tighter algorithm would have a top-down flow control: we would pick some central node to call the "root". Then, nodes would fire in an order depending on their distance to the root. First, every node that is furthest from the root would fire one message. Then, those in the next level would fire one message, etc. This would repeat until we got to the root. The root would have all of its incomming messages, and so could send out all of its outgoing messsges. This would then be true of the nodes a distance 1 from the root, etc. So, the messages would then propogate back down the tree.

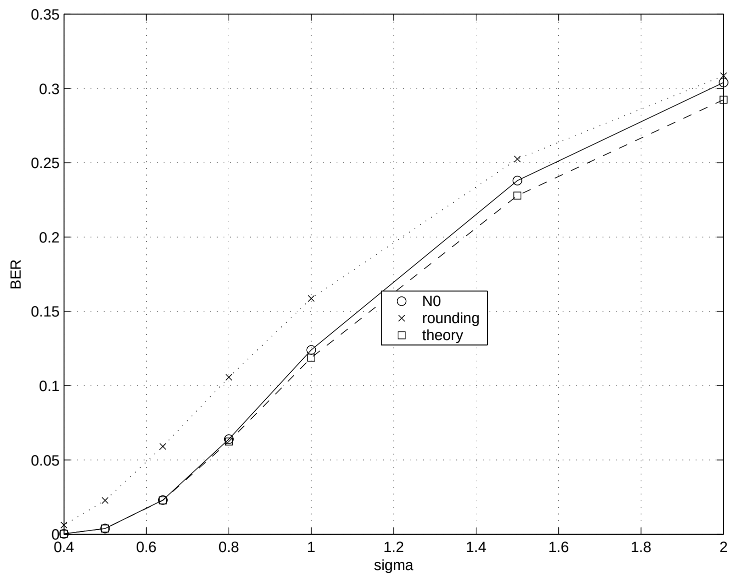
11.4 Infinite trees

What if the tree were infinite? In that case, there would be no nodes of maximum distance from the root. We can make a version of this algorithm work anyway. The idea is to proceed in rounds, and begin by putting a null message incomming on each edge on the first round. That way, each "=" node will have a null message on every edge in the first round, where a null message means half probability zero and half probability one. That way, each "=" node will send an outgoing message on each of its edges. In the next round, this will mean that each "+" node has an incomming message, and so will send an outgoing message. Then, each "=" has an incomming message, etc. If we let this go for k rounds, then the messages being sent out on the external edges will look like the messages we would have created using the tree algorithm on the tree of depth k surrounding that edge.

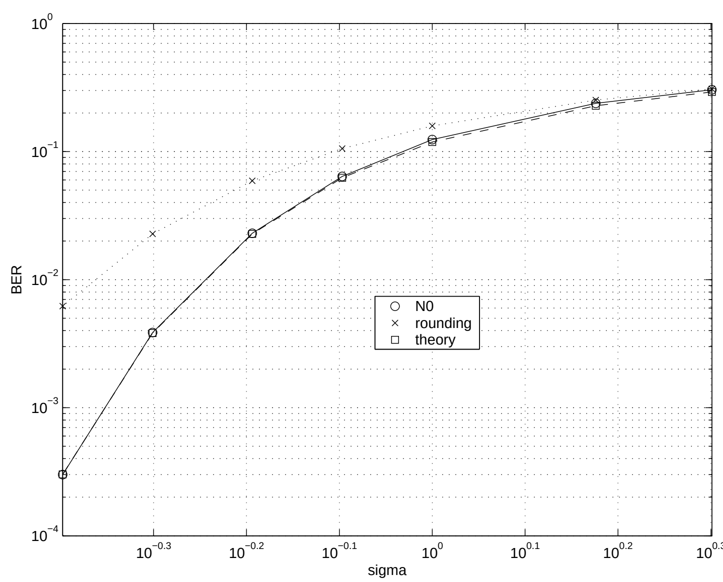
11.5 Small Project 2

You should verify for yourself that this tree algorithm is exactly what we implemented in small project 2, and that you could have sped it up by using the dymanic programming idea.

During class yesterday, Boris observed that the naive decoding algorithm didn't do all that much better than just rounding the bits, at least for large σ. So that you can see this, here are three plots: one for the bit error probability of just rounding the received signal (dotted line with circles), one for the class N0 (solid line with circles), and a theoretical curve I will explain in a moment (dashed line with squares).



It becomes clearer that the coding really is achieving something if you look at the log-log chart:



In particular, the two curves have different slopes.

The theoretical curve is an attempt to explain the performance of the naive algorithm, up to the first order. To do this, let p denote the probability of an error after rounding. If there is only one error, it will probably be corrected. If some bit is in error, and there is another error, let's just assume that the algorithm cannot correct the first bit. (although this is a little pessimistic). Thus, we estimate the probability of a bit error to be

$$p(1-(1-p)^8).$$

This is what I've plotted in the theoretical curve. At first, this curve should not be too different from the rounding curve, because $(1-p)^8$ will be quite small, so it will be likely that there is another error. On the other hand, for p small, this error probability is approximately $8p^2$. Thus, it grows much smaller than p for small p, which explains why the slope of the error curve in the log-log chart for the N0 line should be twice the slope for the rounding line.