Spectral Sparsification of Graphs and Approximations of Matrices

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Objective of Sparsification:

Approximate any (weighted) graph by a sparse weighted graph.

Spanners - Preserve Distances [Chew ‘86]

Cut-Sparsifiers – preserve wt of edges leaving every set $S \subseteq V$ [Benczur-Karger ‘96]

$(1 \pm \epsilon)$
Spectral Sparsification [S-Teng]

Approximate any (weighted) graph by a sparse weighted graph.

Graph $G = (V, E, w)$

Quadratic Form

$$
\sum_{(u,v) \in E} w_{u,v} (x(u) - x(v))^2
= x^T L_G x
$$

$L_G$ is Laplacian of $G$
Laplacian Quadratic Form, examples

All edge-weights are 1

$x : \begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}$
Laplacian Quadratic Form, examples

All edge-weights are 1

\[ x^T L_G x = \text{Sum of squares of differences across edges} \]

\[ = 1 \]
Laplacian Quadratic Form, examples

When $x$ is the characteristic vector of a set $S$, sum the weights of edges on the boundary of $S$

$$x^T L_G x = \sum_{(u,v) \in E} w_{u,v}$$

$u \in S$

$v \notin S$
Laplacian Matrices (quick review)

\[ x^T L_G x = \sum_{(u,v) \in E} w_{u,v} (x(u) - x(v))^2 \]

\[ L_G = D_G - A_G \]

Positive semi-definite

If connected, nullspace = Span(1)
Laplacian Matrices

\[ x^T L_G x = \sum_{(u,v) \in E} w_{u,v} (x(u) - x(v))^2 \]

\[ L_G = \sum_{(u,v) \in E} w_{u,v} L_{u,v} \]

E.g. \[ L_{1,2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \]
\[ = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \]
Laplacian Matrices

\[ x^T L_G x = \sum_{(u,v)\in E} w_{u,v} (x(u) - x(v))^2 \]

\[ L_G = \sum_{(u,v)\in E} w_{u,v} L_{u,v} \]

\[ = \sum_{(u,v)\in E} w_{u,v} (b_{u,v} b_{u,v}^T) \]

where \( b_{u,v} = \delta_u - \delta_v \)
Inequalities on Graphs

For graphs \( G = (V, E, w) \) and \( H = (V, F, z) \)

\[ G \preceq H \]

Iff, for all \( x \in R^V \)

\[ x^T L_G x \leq x^T L_H x \]
Inequalities on Graphs

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

$$G \preceq k \cdot H \quad (\text{multiply edge weights by } k)$$

Iff, for all $x \in \mathbb{R}^V$

$$x^T L_G x \leq k \cdot x^T L_H x$$
Approximation

For graphs \( G = (V, E, w) \) and \( H = (V, F, z) \)

\( H \) is an \( \epsilon \)-approximation of \( G \) if

\[
(1 + \epsilon)^{-1} G \preceq H \preceq (1 + \epsilon) G
\]

That is, for all \( x \in R^V \)

\[
\frac{1}{1 + \epsilon} \leq \frac{\sum_{(u,v) \in F} z_{u,v} (x(u) - x(v))^2}{\sum_{(u,v) \in E} w_{u,v} (x(u) - x(v))^2} \leq 1 + \epsilon
\]
Implications of Approximation

\[(1 + \epsilon)^{-1} G \preceq H \preceq (1 + \epsilon) G\]

\(L_H\) and \(L_G\) have similar eigenvalues

\[(1 + \epsilon)^{-1} x^T L_H^+ x \leq x^T L_G^+ x \leq (1 + \epsilon) x^T L_H^+ x\]

Effective resistances between vertices similar.

\(L_H\) is a good preconditioner for \(L_G\)
Spectral Sparsification [S-Teng]

For an input graph $G$ with $n$ vertices,

find a sparse graph $H$ having $\tilde{O}(n)$ edges

so that $H$ is an $\epsilon$-approximation of $G$
Expander Approximate Complete Graphs

Strong expanders:

d-regular graphs on n vertices

for \( x \perp 1, \quad x^T L_H x \sim dx^T x \)
Expanders Approximate Complete Graphs

For $H$ a $d$-regular strong expander,

For $x \perp 1, \|x\| = 1 \quad x^T L_H x \sim d$

For $G$ the complete graph on $n$ verts.
all non-zero eigs of $L_G$ are $n$.

For $x \perp 1, \|x\| = 1 \quad x^T L_G x = n$
Expanders Approximate Complete Graphs

For $H$ a $d$-regular strong expander,

For $x \perp 1$, $\|x\| = 1$ \quad $x^T L_H x \sim d$

For $G$ the complete graph on $n$ verts.

all non-zero eigs of $L_G$ are $n$.

For $x \perp 1$, $\|x\| = 1$ \quad $x^T L_G x = n$

$\frac{n}{d} H$ is a good approximation of $G$
Best Approximations of Complete Graphs

Ramanujan Expanders
[Margulis, Lubotzky-Phillips-Sarnak]

\[ d - 2\sqrt{d - 1} \leq \lambda(L_H) \leq d + 2\sqrt{d - 1} \]
Best Approximations of Complete Graphs

Ramanujan Expanders
[Margulis, Lubotzky-Phillips-Sarnak]

\[ d - 2\sqrt{d - 1} \leq \lambda(L_H) \leq d + 2\sqrt{d - 1} \]

Can we approximate every graph this well?
Example: Dumbbell

G
Complete graph on n vertices

H
d-regular Ramanujan, times n/d

1

Complete graph on n vertices
d-regular Ramanujan, times n/d
Example: Dumbbell

\[ \text{K}_n \quad 1 \quad \text{K}_n \]

\[ \text{d-regular Ramanujan, times n/d} \quad 1 \quad \text{d-regular Ramanujan, times n/d} \]
Example: Dumbbell

\[ G = G_1 + G_2 + G_3 \]
\[ H = H_1 + H_2 + H_3 \]

\( G_1 \leq (1 + \epsilon)H_1 \)
\( G_2 = H_2 \)
\( G_3 \leq (1 + \epsilon)H_3 \)
Main Theorem

For every $G = (V, E, w)$, there is a $H = (V, F, z)$ s.t.

1. $H$ is an $\varepsilon$ -approximation of $G$

2. $|F| \leq |V| \frac{(2 + \varepsilon)^2}{\varepsilon^2}$ (Batson-S-Srivastava 09)

3. $F \subseteq E$
Main Theorem (today)

For every $G = (V, E, w)$, there is a $H = (V, F, z)$ s.t.

1. H is a 2.6-approximation of $G$

2. $|F| \leq 6|V|$ (Batson-S-Srivastava 09)

3. $F \subseteq E$
Reduction to Matrix Approximation

\[ 1 \leq \frac{x^T L_H x}{x^T L_G x} \leq 13 \quad \forall x \perp 1 \]

Setting \( x = L_G^{-1/2} z \)

Becomes \( 1 \leq \frac{z^T L_G^{-1/2} L_H L_G^{-1/2} z}{z^T z} \leq 13 \quad \forall z \perp 1 \)
Reduction to Matrix Approximation

Suffices to show

\[ 1 \leq \frac{z^T L_G^{-1/2} L_H L_G^{-1/2} z}{z^T z} \leq 13 \quad \forall z \perp 1 \]

\[ 1 \leq \lambda \left( L_G^{-1/2} L_H L_G^{-1/2} \right) \leq 13 \]
Reduction to Matrix Approximation

\[ 1 \leq \lambda \left( L_G^{-1/2} L_H L_G^{-1/2} \right) \leq 13 \]

Write \( L_H = \sum s_e (b_e b_e^T) \)

weight of edge \( e \) in graph \( H \)

Laplacian of edge \( e \), as outer-product of vectors

Need \[ 1 \leq \lambda \left( \sum_e s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2} \right) \leq 13 \]

At most \( 6n \) of the \( s_e \) non-zero
Reduction to Matrix Approximation

Need

\[ 1 \leq \lambda \left( \sum_{e} s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2} \right) \leq 13 \]

At most $6n$ of the $s_e$ non-zero
Reduction to Matrix Approximation

Need \[ 1 \leq \lambda \left( \sum_e s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2} \right) \leq 13 \]

At most \(6n\) of the \(s_e\) non-zero

Recall \[ L_G = \sum_e b_e b_e^T \]

So \[ \sum_e L_G^{-1/2} b_e b_e^T L_G^{-1/2} = I \]
Reduction to Matrix Approximation

Need \( 1 \leq \lambda(\sum_{e} s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2}) \leq 13 \)

At most \( 6n \) of the \( s_e \) non-zero

Set \( v_e = L_G^{-1/2} b_e \)

\[
1 \leq \lambda(\sum_{e} s_e v_e v_e^T) \leq 13
\]

\[
\sum_{e} v_e v_e^T = \sum_{e} L_G^{-1/2} b_e b_e^T L_G^{-1/2} = L_G^{-1/2} L_G L_G^{-1/2} = id
\]
A closer look at $\nu_e$

$v_e = L_G^{-1/2} b_e.$

$\sum_{e} v_e v_e^T = I$

“decomposition of identity”

$m$ vectors in $\mathbb{R}^{n-1}$
A closer look at $v_e$

$v_e = L_G^{-1/2} b_e.$

$\sum_e v_e v_e^T = I$

"decomposition of identity"
A closer look at $\mathbf{v}_e$

$m$ vectors in $\mathbb{R}^{n-1}$

$\forall u \quad \sum_e \langle u, \mathbf{v}_e \rangle^2 = 1$
Choosing a Subgraph

\( G \rightarrow H \)
New Goal

$\forall u : 1 \leq \sum_e s_e \langle u, v_e \rangle^2 \leq 13$
Existence theorem

If
\[ \sum_{e} v_{e} v_{e}^T = I_{n} \]
then there are scalars \( s_{e} \geq 0 \) with
\[ 1 \leq \lambda(\sum_{e} s_{e} v_{e} v_{e}^T) \leq 13 \]
and \( |\{ s_{e} \neq 0 \}| \leq 6n. \)
Existence theorem

If

\[ \sum_{e} v_e v_e^T = I_n \]
then there are scalars \( s_e \geq 0 \) with

\[ 1 \leq \lambda(\sum_{e} s_e v_e v_e^T) \leq \frac{d + 1 + 2\sqrt{d}}{d + 1 - 2\sqrt{d}} \]

and \( |\{s_e \neq 0\}| \leq dn \)
Existence theorem

If

\[ \sum_{e} v_{e} v_{e}^{T} = I_{n} \]

then there are scalars \( s_{e} \geq 0 \) with

\[ 1 \leq \lambda(\sum_{e} s_{e} v_{e} v_{e}^{T}) \leq 13 \]

and \( |\{s_{e} \neq 0\}| \leq 6n \).
What happens when we add a vector?

\[ \lambda(A) \]
Interlacing

\[ \lambda(A) \]

\[ \lambda(A + vv^T) \]
More precisely

Characteristic Polynomial:

$$p_A(x) = \det(xI - A)$$
More precisely

Characteristic Polynomial:

\[ p_A(x) = \det(xI - A) \]

Rank-one update:

\[
p_{A+vv^T} = \left(1 + \sum_i \frac{<v,u_i>^2}{\lambda_i - x} \right) p_A
\]

Where \( Au_i = \lambda_i u_i \)
More precisely

Characteristic Polynomial:

$$p_A(x) = \det(xI - A)$$

Rank-one update:

$$p_{A+vv^T} = \left(1 + \sum_i \frac{\langle v, u_i \rangle^2}{\lambda_i - x} \right) p_A$$

\(\lambda(A + vv^T)\) are zeros of
Physical model of interlacing

\( \lambda_i = \text{positive unit charges resting at barriers on a slope} \)
Physical model of interlacing

$$\lambda(A + vv^T)$$

$u_i$ is eigenvector
$v$ is added vector
$\langle v, u_i \rangle^2$ charge on barrier
Physical model of interlacing

Barriers repel eigs.

\( + \langle v, u_n \rangle^2 \)

\( + \langle v, u_2 \rangle^2 \)

\( + \langle v, u_1 \rangle^2 \)

\( \lambda_1 \)

\( \lambda_2 \)

\( u_i \) is eigenvector

\( v \) is added vector

\( \langle v, u_i \rangle^2 \) charge on barrier
Physical model of interlacing

Barriers repel eigs.

Inverse law repulsion

Gravity

\[ 1 + \sum_i \frac{\langle v, u_i \rangle^2}{\lambda_i - x} = 0 \]
Examples
Ex1: All weight on $u_1$
Ex1: All weight on $u_1$
Ex1: All weight on $u_1$
Ex1: All weight on $u_1$

$\nu = u_1$

Gravity keeps resting on barrier

Pushed up against next barrier

$\lambda(A + \nu\nu^T)$
Ex2: Equal weight on $u_1, u_2$
Ex2: Equal weight on $u_1, u_2$
Ex2: Equal weight on $u_1, u_2$

$$\lambda(\mathbf{A} + \mathbf{v}\mathbf{v}^T)$$
Ex3: Equal weight on all $u_1, u_2, \ldots, u_n$
Ex3: Equal weight on all $u_1, u_2, \ldots, u_n$
Adding a random $v_e$

Because $v_e$ are decomposition of identity,

$$\mathbb{E}_{e} \left[ \langle v_e, u_i \rangle^2 \right] = 1/m$$

$$\mathbb{E}_{e} \left[ P_{A + v_e v^T_e} \right] = \left( 1 + \frac{1}{m} \sum_i \frac{1}{\lambda_i - x} \right) P_A$$

$$= P_A - \frac{1}{m} \frac{d}{dx} P_A$$
Ideal proof

\[ A^{(0)} = 0 \]
\[ p^{(0)} = x^n \]
Ideal proof

\[ E_e \langle v_e, u_i \rangle^2 = \frac{1}{m} \]

\[ A^{(0)} = 0 \]

\[ p^{(0)} = x^n \]
$p^{(1)} = x^n - \frac{1}{m} \frac{d}{dx} x^n$

$A^{(1)} = 0 + v v^T$

$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$

Ideal proof
Ideal proof

\[ A^{(2)} = A^{(1)} + vv^T \]

\[ p^{(2)} = (1 - \frac{1}{m} \frac{d}{dx})^2 x^n \]

\[ \mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m \]
Ideal proof

\[ \mathbb{E}_e \langle v_e, u_i \rangle^2 = \frac{1}{m} \]

\[ A(i+1) = A(i) + \nu \nu^T \]

\[ p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n \]
Ideal proof

\[ A^{(i+1)} = A^{(i)} + \nu \nu^T \]

\[ p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n \]
Ideal proof

\[ A(i+1) = A(i) + vv^T \]

\[ p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n \]
Ideal proof

\[ A_{i+1} = A_i + vv^T \]

\[ p^{(i+1)} = \left( 1 - \frac{1}{m} \frac{d}{dx} \right)^{i+1} x^n \]
Ideal proof

\[
\mathbb{E} e \langle v_e, u_i \rangle^2 = 1/m
\]

\[
A^{(i+1)} = A^{(i)} + \nu \nu^T
\]

\[
p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n
\]
Ideal proof

$$A^{(i+1)} = A^{(i)} + \nu \nu^T$$

$$p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n$$

$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = \frac{1}{m}$$
Ideal proof

\[
\langle v_e, u_i \rangle^2 = \frac{1}{m}
\]

\[
A(i+1) = A(i) + vv^T
\]

\[
p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n
\]
Ideal proof

\[ A^{(i+1)} = A^{(i)} + \nu \nu^T \]

\[ p^{(i+1)} = \left( 1 - \frac{1}{m} \frac{d}{dx} \right)^{i+1} x^n \]
$$E_e \langle v_e, u_i \rangle^2 = \frac{1}{m}$$

$$A(i+1) = A(i) + v v^T$$

$$p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n$$

$$\frac{\lambda_n(A)}{\lambda_1(A)} \leq 13?$$
Ideal proof

\[ A(i+1) = A(i) + vv^T \]

\[ p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n \]

\[ \frac{\lambda_n(A)}{\lambda_1(A)} \rightarrow 1 \]

= associated Laguerre polynomial
Broad outline: moving barriers

\[ A = \emptyset \]
Step 1

\[ A = \emptyset \]

\[ +\mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\} \]
Step 1

\[ A = \emptyset \]

\[ +vv^T \quad v \in \{v_e\} \]

\[ A = vv^T \]
Step 1

\[ A = \emptyset \]

\[ +vv^T \quad v \in \{v_e\} \]

\[ A = vv^T \]

\[ + \frac{1}{3} \quad - \frac{1}{3} \quad +2 \]
Step 1

\[ A = \emptyset \]

-\( n \)

+1/3

-\( n+1/3 \)

0

\[ A = vv^T \quad v \in \{v_e\} \]

+2

n+2

looser constraint

tighter constraint
Step $i+1$

$A^{(i)}$

$0$

$\leq \lambda_i \leq$
Step $i+1$

$A(i)$

$\text{+} v v^T$
Step $i+1$

$A(i), A(i+1)$

$\leq \lambda_i \leq$
Step \(i+1\)

\[ A(i), A(i+1) \]

\[ +\frac{1}{3} \]

\[ +2 \]

\[ 0 \]

\[ +vv^T \]
Step $i+1$

$A(i), A(i+1), A(i+2)$

$0$

$\leq \lambda_i \leq$
Step $i+1$

$A(i), A(i+1), A(i+2)$

$\mathbf{0} + \mathbf{v} \mathbf{v}^T$
Step \( i+1 \)

\[ A(i), A(i+1), A(i+2), A(i+3) \]
Step $i+1$

$A(i), A(i+1), A(i+2), A(i+3), \ldots$

$0$

$\leq \lambda_i \leq$
Step \(i+1\)

\[ A(i), A(i+1), A(i+2), A(i+3), \ldots \]
Step $i+1$

$A(i), A(i+1), A(i+2), A(i+3), \ldots$

$0$

$\leq \lambda_i \leq$
Step 6n

\[ A(i), A(i+1), A(i+2), A(i+3), \ldots, A(6n) \]
Step 6n

\[ A(i), A(i+1), A(i+2), A(i+3), \ldots, A(6n) \]

2.6-approximation with 6n vectors.
Problem

need to show that an appropriate $v_e v_e^T$ always exists.
Problem

need to show that an appropriate
\[ v e v_e^T \]
always exists.

\[ \leq \lambda_i \leq \]

Is not strong enough for induction
Problems

If many small eigs, can only move one

If $v_e$ has large inner product with large eigenvectors, the largest eigenvalue moves far.
The Lower Barrier Potential Function

\[ \Phi_\ell(A) = \sum_i \frac{1}{\lambda_i - \ell} = \text{Tr} \left( (A - \ell I)^{-1} \right) \]
The Lower Barrier Potential Function

\[ \Phi_\ell(A) = \sum_i \frac{1}{\lambda_i - \ell} = \text{Tr} \left( (A - \ell I)^{-1} \right) \]

\[ \Phi_\ell(A) \leq 1 \implies \lambda_{min}(A) \geq \ell + 1 \]
The Lower Barrier Potential Function

$$\Phi_\ell(A) = \sum_i \frac{1}{\lambda_i - \ell} = \text{Tr} \left( (A - \ell I)^{-1} \right)$$

No $\lambda_i$ within dist. 1
No 2 $\lambda_i$ within dist. 2
No 3 $\lambda_i$ within dist. 3
... No $k \lambda_i$ within dist. $k$

$$\Phi_\ell(A) \leq 1 \implies \lambda_{\text{min}}(A) \geq \ell + 1$$
The Upper Barrier Potential Function

\[ \Phi^u(A) = \sum_i \frac{1}{u - \lambda_i} = \text{Tr} \left( (uI - A)^{-1} \right) \]
The Beginning

\[ A = \emptyset \]
The Beginning

\[ A = \emptyset \]

\[ \phi^n(\emptyset) = \text{Tr}(nI)^{-1} = 1 \]

\[ \phi_{-n}(\emptyset) = \text{Tr}(nI)^{-1} = 1. \]
Step $i+1$

$A(i)$, $A(i+1)$, $A(i+2)$

$\Phi^u(A) \leq 1$

$\Phi_\ell(A) \leq 1$. 
Lemma. can always choose so that potentials do not increase

\[ \Phi^u(A) \leq 1 \]

\[ \Phi^\ell(A) \leq 1. \]
Step $i+1$

$A(i), A(i+1), A(i+2), A(i+3)$

$\Phi^u(A) \leq 1$

$\Phi_\ell(A) \leq 1.$
Step $i+1$

$A(i)$, $A(i+1)$, $A(i+2)$, $A(i+3)$, ...

$\Phi^u(A) \leq 1$

$\Phi^\ell(A) \leq 1.$
Step $i+1$

$A(i), A(i+1), A(i+2), A(i+3), \ldots$

$\Phi^u(A) \leq 1$

$\Phi^\ell(A) \leq 1.$
Step $i+1$

$A(i), A(i+1), A(i+2), A(i+3), \ldots$

$0$

$\Phi^u(A) \leq 1$

$\Phi_\ell(A) \leq 1$.
Step 6n

$A(i), A(i+1), A(i+2), A(i+3), \ldots, A(6n)$

$\Phi^u(A) \leq 1$

$\Phi_\ell(A) \leq 1$
Step 6n

$A(i), A(i+1), A(i+2), A(i+3), \ldots, A(6n)$

2.6-approximation with 6n vectors.
Goal

Lemma. can always choose $+ s \mathbf{v} \mathbf{v}^T \Phi_u(A) \leq 1$
so that potentials do not increase $\Phi_\ell(A) \leq 1$. 

$+ 1/3$ $+ 2$ 
$+ s \mathbf{v} \mathbf{v}^T$
Upper Barrier Update

Add $svv^T$ and set $u' \leftarrow u + 2$. 
Upper Barrier Update

Add \( svv^T \) and set \( u' \leftarrow u + 2. \)

\[
\Phi^{u'}(A + svv^T) = \text{Tr} \left( (u'I - A - svv^T)^{-1} \right)
\]
Add $s v v^T$ and set $u' \leftarrow u + 2$.

$$
\Phi_{u'}(A + s v v^T)
= \text{Tr} \left( \left(u' I - A - s v v^T \right)^{-1} \right)
= \Phi_{u'}(A) + \frac{s v^T (u' I - A)^{-2} v}{1 - s v^T (u' I - A)^{-1} v}
$$

By Sherman-Morrison Formula
Upper Barrier Update

Add \( s \nu \nu^T \) and set \( u' \leftarrow u + 2 \).

\[
\Phi^u' (A + s \nu \nu^T)
\]

\[
= \text{Tr} \left( (u'I - A - s \nu \nu^T)^{-1} \right)
\]

\[
= \Phi^u' (A) + \frac{s \nu^T (u'I - A)^{-2} \nu}{1 - s \nu^T (u'I - A)^{-1} \nu}
\]

Need \( \leq \Phi^u (A) \)
How much of $\mathbf{v} \mathbf{v}^T$ can we add?

Rearranging:

$$\Phi^u(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^u(A)$$

iff

$$1 \geq s\mathbf{v}^T \left( \frac{(u' I - A)^{-2}}{\Phi^u(A) - \Phi^{u'}(A)} + (u' I - A)^{-1} \right) \mathbf{v}$$
How much of $\mathbf{vv}^T$ can we add?

Rearranging:

$$\Phi^u'(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^u(A)$$

iff

$$1 \geq s\mathbf{v}^T \left( \frac{(u'I - A)^{-2}}{\Phi^u(A) - \Phi^u'(A)} + (u'I - A)^{-1} \right) \mathbf{v}$$

Write as

$$1 \geq s\mathbf{v}^T U_A \mathbf{v}$$
Lower Barrier

Similarly:

\[ \Phi_{l'}(A + svv^T) \leq \Phi_l(A) \]

iff

\[ 1 \leq sv^T \left( \frac{(A - l' I)^{-2}}{\Phi_{l'}(A) - \Phi_l(A)} - (A - l' I)^{-1} \right) v \]

Write as

\[ 1 \leq sv^T L_A v \]
Goal

Show that we can always add some vector while respecting both barriers.

Need: \( sv^T U_A v \leq 1 \leq sv^T L_A v \)
Two expectations

Need: \( sv^T U_A v \leq 1 \leq sv^T L_A v \)

Can show: \( E_e \left[ v_e^T U_A v_e \right] \leq 1 \)
\( E_e \left[ v_e^T L_A v_e \right] \geq 1 \)
Two expectations

Need: \( s \mathbf{v}^T U_A \mathbf{v} \leq 1 \leq s \mathbf{v}^T L_A \mathbf{v} \)

Can show: \( E_e \left[ \mathbf{v}_e^T U_A \mathbf{v}_e \right] \leq 1 \)
\( E_e \left[ \mathbf{v}_e^T L_A \mathbf{v}_e \right] \geq 1 \)

So: \( E_e \left[ \mathbf{v}_e^T U_A \mathbf{v}_e \right] \leq E_e \left[ \mathbf{v}_e^T L_A \mathbf{v}_e \right] \)

And, exists \( e : \mathbf{v}_e^T U_A \mathbf{v}_e \leq \mathbf{v}_e^T L_A \mathbf{v}_e \)
Two expectations

Need: \( sv^T U_A v \leq 1 \leq sv^T L_A v \)

Can show: \( E_e \left[ v_e^T U_A v_e \right] \leq 1 \)
\( E_e \left[ v_e^T L_A v_e \right] \geq 1 \)

So: \( E_e \left[ v_e^T U_A v_e \right] \leq E_e \left[ v_e^T L_A v_e \right] \)

And, exists \( e : v_e^T U_A v_e \leq v_e^T L_A v_e \)

And \( s \) that puts 1 between them
Bounding expectations

\[ \nu^T U_A \nu = \text{Tr} \left( U_A \nu \nu^T \right) \]

\[ \mathbb{E}_e \left[ \text{Tr} \left( U_A \nu_e \nu_e^T \right) \right] = \text{Tr} \left( U_A \mathbb{E}_e \left[ \nu_e \nu_e^T \right] \right) \]

\[ = \text{Tr} \left( U_A I \right) \]

\[ = \text{Tr} \left( U_A \right) \]
Bounding expectations

\[
\text{Tr} (U_A) = \frac{\text{Tr} \left( (u' \, I - A)^{-2} \right)}{\Phi u (A) - \Phi u' (A)} + \text{Tr} \left( (u' \, I - A)^{-1} \right)
\]

\[
\leq \frac{1}{u' - u} \leq \frac{1}{2} + 1 = \frac{3}{2}
\]

\[
= \Phi u' (A)
\]
Bounding expectations

Similarly,

$$\text{Tr} \left( L_A \right) \geq \frac{1}{l' - l} - 1$$

$$= \frac{1}{1/3} - 1$$

$$= 2$$

So

$$\mathbf{E}_e \left[ \mathbf{v}_e^T U_A \mathbf{v}_e \right] = \text{Tr} \left( U_A \right) \leq \text{Tr} \left( L_A \right) = \mathbf{E}_e \left[ \mathbf{v}_e^T L_A \mathbf{v}_e \right]$$
Lemma. can always choose so that potentials do not increase

\[ A(i), A(i+1), A(i+2) \]

\[ + \frac{1}{3}, +2 \]

\[ \Phi^u(A) \leq 1 \]

\[ \Phi_\ell(A) \leq 1. \]
Step $i+1$

$A(i), A(i+1), A(i+2), A(i+3)$

$\Phi^u(A) \leq 1$

$\Phi^\ell(A) \leq 1$. 
Step $i+1$

$A(i), A(i+1), A(i+2), A(i+3), \ldots$

$\Phi^u(A) \leq 1$

$\Phi^\ell(A) \leq 1$. 
Step $i+1$

$A(i), A(i+1), A(i+2), A(i+3), \ldots$

$\Phi^u(A) \leq 1$

$\Phi_\ell(A) \leq 1$. 
Step $i+1$

$A(i), A(i+1), A(i+2), A(i+3), \ldots$

$\Phi^u(A) \leq 1$
$\Phi^\ell(A) \leq 1.$
Step 6n

$A(i), A(i+1), A(i+2), A(i+3), \ldots, A(6n)$

$\Phi_u (A) \leq 1$

$\Phi_\ell (A) \leq 1$.
Step 6n

\[ A(i), A(i+1), A(i+2), A(i+3), \ldots, A(6n) \]

2.6-approximation with 6n vectors.
Twice-Ramanujan

Fixing $dn$ steps and tightening parameters gives ratio

$$\frac{\lambda_{max}(A)}{\lambda_{min}(A)} \leq \frac{d + 1 + 2\sqrt{d}}{d + 1 - 2\sqrt{d}}$$

Less than twice as many edges as used by Ramanujan Expander of same quality
Newman-Rabinovich:
  Cut dimension of $\ell_1$-metrics

Schechtman:
  Embedding $k$-dim subspaces of $L_p$ into $\ell_p^n$

Nitzan-Olevskii-Ulanovskii:
  Sampling sequences for Paley-Wiener space

S-Srivastava:
  Bourgain-Tzafriri Restricted Invertibility

Srivastava-Vershynin:
  Better concentration for random matrices
Open Questions

- Unweighted sparsifiers of complete graph.
- The Ramanujan bound
- Properties of vectors from graphs?
- Faster algorithm union of random Hamiltonian cycles?
- The Kadison-Singer Conjecture
Thank you!