Ramanujan Graphs of Every Degree

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Expander Graphs

Sparse, regular well-connected graphs with many properties of random graphs.

Random walks mix quickly.
Every set of vertices has many neighbors.
Pseudo-random generators.
Error-correcting codes.
Sparse approximations of complete graphs.
Spectral Expanders

Let $G$ be a graph and $A$ be its adjacency matrix

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 \\
\end{pmatrix}
$$

Eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

If $d$-regular (every vertex has $d$ edges), $\lambda_1 = d$
Spectral Expanders

If bipartite (all edges between two parts/colors) eigenvalues are symmetric about 0

If $d$-regular and bipartite, $\lambda_n = -d$

```
0 0 0 1 0 1
0 0 0 1 1 0
0 0 0 0 1 1
1 1 0 0 0 0
0 1 1 0 0 0
1 0 1 0 0 0
```
Spectral Expanders

$G$ is a good spectral expander if all non-trivial eigenvalues are small.
Bipartite Complete Graph

Adjacency matrix has rank 2, so all non-trivial eigenvalues are 0

```
 0 0 0 1 1 1
 0 0 0 1 1 1
 0 0 0 1 1 1
 1 1 1 0 0 0
 1 1 1 0 0 0
 1 1 1 0 0 0
```
Spectral Expanders

$G$ is a good spectral expander if all non-trivial eigenvalues are small.

Challenge:
construct infinite families of fixed degree
Spectral Expanders

$G$ is a good spectral expander if all non-trivial eigenvalues are small.

\[ -d \quad -2\sqrt{d-1} \quad 0 \quad 2\sqrt{d-1} \quad d \]

Challenge:
construct infinite families of fixed degree

Alon-Boppana ‘86: Cannot beat $2\sqrt{d-1}$
Ramanujan Graphs: \(2\sqrt{d - 1}\)

\
\[ G \text{ is a Ramanujan Graph if absolute value of non-trivial eigs } \leq 2\sqrt{d - 1} \]

-\(d\) \(\rightarrow\) \(-2\sqrt{d - 1}\) \(\rightarrow\) 0 \(\rightarrow\) \(2\sqrt{d - 1}\) \(\rightarrow\) \(d\)
Ramanujan Graphs: \(2\sqrt{d - 1}\)

\[G\] is a Ramanujan Graph
if absolute value of non-trivial eigs \(\leq 2\sqrt{d - 1}\)

Margulis, Lubotzky-Phillips-Sarnak'88: Infinite sequences of Ramanujan graphs exist for \(d = \text{prime} + 1\)
Ramanujan Graphs: \( 2\sqrt{d} - 1 \)

\( G \) is a Ramanujan Graph if absolute value of non-trivial eigs \( \leq 2\sqrt{d} - 1 \)

\[ -d \quad -2\sqrt{d - 1} \quad 0 \quad 2\sqrt{d - 1} \quad d \]

Friedman’08: A random \( d \)-regular graph is almost Ramanujan: \( 2\sqrt{d - 1} + \epsilon \)
Theorem:
there are infinite families of bipartite Ramanujan graphs of every degree.
Theorem:
there are infinite families of bipartite Ramanujan graphs of every degree.

And, are infinite families of \((c,d)\)-biregular Ramanujan graphs, having non-trivial eigs bounded by

\[\sqrt{d - 1} + \sqrt{c - 1}\]
Bilu-Linial ‘06 Approach

Find an operation that doubles the size of a graph without creating large eigenvalues.

\[
\begin{align*}
-d & \quad ( & \quad 0 & \quad ) & \quad 2\sqrt{d-1} \\
-2\sqrt{d-1} & \quad 0 & \quad 2\sqrt{d-1} & \quad d
\end{align*}
\]
Bilu-Linial ‘06 Approach

Find an operation that doubles the size of a graph without creating large eigenvalues.
2-lifts of graphs
2-lifts of graphs

duplicate every vertex
2-lifts of graphs

duplicate every vertex
2-lifts of graphs

for every pair of edges: leave on either side (parallel), or make both cross
2-lifts of graphs

for every pair of edges: leave on either side (parallel), or make both cross
2-lifts of graphs

\[
\begin{array}{cccccc}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
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\end{array}
\]
2-lifts of graphs

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2-lifts of graphs

\[
\begin{align*}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
\end{align*}
\]
Eigenvalues of 2-lifts (Bilu-Linial)

Given a 2-lift of $G$, create a signed adjacency matrix $A_s$ with a -1 for crossing edges and a 1 for parallel edges.

\[
\begin{pmatrix}
0 & -1 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{pmatrix}
\]
Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:
The eigenvalues of the 2-lift are the union of the eigenvalues of $A$ (old) and the eigenvalues of $A_s$ (new)

\[
\begin{pmatrix}
0 & -1 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 \\
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1 & 1 & 0 & 1 & 0 \\
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Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:
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Conjecture:
Every $d$-regular graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d-1}$
Eigenvalues of 2-lifts (Bilu-Linial)

Conjecture:
Every $d$-regular graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d - 1}$

Would give infinite families of Ramanujan Graphs:

start with the complete graph, and keep lifting.
Eigenvalues of 2-lifts (Bilu-Linial)

Conjecture:
Every $d$-regular graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d-1}$

We prove this in the bipartite case.

A 2-lift of a bipartite graph is bipartite.
Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:
Every $d$-regular graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d - 1}$.

Trick: eigenvalues of bipartite graphs are symmetric about 0, so only need to bound largest.
Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:
Every $d$-regular \textbf{bipartite} graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d-1}$.
First idea: a random 2-lift

Specify a lift by \( s \in \{\pm 1\}^m \)

Pick \( s \) uniformly at random
First idea: a random 2-lift

Specify a lift by \( s \in \{\pm 1\}^m \)

Pick \( s \) uniformly at random

Are graphs for which this usually fails
First idea: a random 2-lift

Specify a lift by $s \in \{\pm 1\}^m$

Pick $s$ uniformly at random

Are graphs for which this usually fails

Bilu and Linial proved $G$ almost Ramanujan, implies new eigenvalues usually small.

Improved by Puder and Agarwal-Kolla-Madan
The expected polynomial

Consider \( E_s \left[ \chi A_s (x) \right] \)
The expected polynomial

Consider \( \mathbb{E}_s \left[ \chi_{A_s}(x) \right] \)

Prove max-root \( \left( \mathbb{E}_s \left[ \chi_{A_s}(x) \right] \right) \leq 2\sqrt{d-1} \)

Prove \( \chi_{A_s}(x) \) is an interlacing family

Conclude there is an \( s \) so that \( \text{max-root} \left( \chi_{A_s}(x) \right) \leq 2\sqrt{d-1} \)
The expected polynomial

Theorem (Godsil-Gutman ’81):

$$\mathbb{E}_s \left[ \chi A_s (x) \right] = \mu_G (x)$$

the matching polynomial of $G$
The matching polynomial
(Heilmann-Lieb ‘72)

\[ \mu_G(x) = \sum_{i \geq 0} x^{n-2i} (-1)^i m_i \]

\[ m_i = \text{the number of matchings with } i \text{ edges} \]
\[ \mu_G(x) = x^6 - 7x^4 + 11x^2 - 2 \]
\[ \mu_G(x) = x^6 - 7x^4 + 11x^2 - 2 \]

One matching with 0 edges
\[ \mu_G(x) = x^6 - 7x^4 + 11x^2 - 2 \]

7 matchings with 1 edge
\[ \mu_G(x) = x^6 - 7x^4 + 11x^2 - 2 \]
\[
\mu_G(x) = x^6 - 7x^4 + 11x^2 - 2
\]
Proof that \[ \mathbb{E}_s \left[ \chi A_s(x) \right] = \mu_G(x) \]

Expand \[ \mathbb{E}_s \left[ \det(xI - A_s) \right] \] using permutations

\[
\begin{array}{cccccc}
  x & \pm1 & 0 & 0 & \pm1 & \pm1 \\
\pm1 & x & \pm1 & 0 & 0 & 0 \\
0 & \pm1 & x & \pm1 & 0 & 0 \\
0 & 0 & \pm1 & x & \pm1 & 0 \\
\pm1 & 0 & 0 & \pm1 & x & \pm1 \\
\pm1 & 0 & 0 & 0 & \pm1 & x \\
\end{array}
\]
Proof that $\mathbb{E}_s[\chi_{A_s}(x)] = \mu_G(x)$

Expand $\mathbb{E}_s[\det(xI - A_s)]$ using permutations

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same edge: x, ±1
same value: ±1, 0
Proof that \( \mathbb{E}_s \left[ \chi_{A_s}(x) \right] = \mu_G(x) \)

Expand \( \mathbb{E}_s \left[ \det(xI - A_s) \right] \) using permutations

same edge: \( x \pm 1 \quad 0 \quad 0 \quad \pm 1 \quad \pm 1 \)
\( \pm 1 \quad x \quad \pm 1 \quad 0 \quad 0 \quad 0 \)
\( 0 \quad \pm 1 \quad x \quad \pm 1 \quad 0 \quad 0 \)
\( 0 \quad 0 \quad \pm 1 \quad x \quad \pm 1 \quad 0 \)
\( \pm 1 \quad 0 \quad 0 \quad \pm 1 \quad x \quad \pm 1 \)
\( \pm 1 \quad 0 \quad 0 \quad 0 \quad \pm 1 \quad x \)

same value
Proof that $\mathbb{E}_s [ \chi_{A_s}(x) ] = \mu_G(x)$

Expand $\mathbb{E}_s [ \det(xI - A_s) ]$ using permutations

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Get 0 if hit any 0s
Proof that $\mathbb{E}_s \left[ \chi_{A_s}(x) \right] = \mu_G(x)$

Expand $\mathbb{E}_s \left[ \det(xI - A_s) \right]$ using permutations

Get 0 if take just one entry for any edge
Proof that $\mathbb{E}_s[\chi_{A_s}(x)] = \mu_G(x)$

Expand $\mathbb{E}_s[\det(xI - A_s)]$ using permutations

Only permutations that count are involutions
Proof that $\mathbb{E}_s \left[ \chi_{A_s}(x) \right] = \mu_G(x)$

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Proof that $\mathbb{E}_s [ \chi_{A_s}(x) ] = \mu_G(x)$

Expand $\mathbb{E}_s [ \det(xI - A_s) ]$ using permutations

Only permutations that count are involutions

Correspond to matchings
The matching polynomial
(Heilmann-Lieb ‘72)

\[ \mu_G(x) = \sum_{i \geq 0} x^{n-2i} (-1)^i m_i \]

Theorem (Heilmann-Lieb)
all the roots are real
The matching polynomial
(Heilmann-Lieb ‘72)

\[ \mu_G(x) = \sum_{i \geq 0} x^{n-2i} (-1)^i m_i \]

Theorem (Heilmann-Lieb)
all the roots are real
and have absolute value at most \(2\sqrt{d - 1}\)
The matching polynomial
(Heilmann-Lieb ’72)

\[ \mu_G(x) = \sum_{i \geq 0} x^{n-2i} (-1)^i m_i \]

Theorem (Heilmann-Lieb)
all the roots are real
and have absolute value at most \( \sqrt{d-1} \)

Implies max-root \( \left( \mathbb{E} [ \chi_{A_s}(x) ] \right) \leq 2\sqrt{d-1} \)
Interlacing

Polynomial $p(x) = \prod_{i=1}^{d} (x - \alpha_i)$

interlaces $q(x) = \prod_{i=1}^{d-1} (x - \beta_i)$

if $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \alpha_{d-1} \leq \beta_{d-1} \leq \alpha_d$
$p_1(x)$ and $p_2(x)$ have a common interlacing if can partition the line into intervals so that each interval contains one root from each poly
Common Interlacing

$p_1(x)$ and $p_2(x)$ have a common interlacing if can partition the line into intervals so that each interval contains one root from each poly
Common Interlacing

If $p_1$ and $p_2$ have a common interlacing, 
\[
\max\text{-root} \left( p_i \right) \leq \max\text{-root} \left( \mathbb{E}_i \left[ p_i \right] \right)
\]
for some $i$. 

Largest root of average
If $p_1$ and $p_2$ have a common interlacing, 
\[ \text{max-root} \left( p_i \right) \leq \text{max-root} \left( \mathbb{E}_i \left[ p_i \right] \right) \]
for some $i$. 

**Common Interlacing**

Largest root of average
Interlacing Family of Polynomials

$$\{p_s\}_{s \in \{\pm 1\}^m} \text{ is an interlacing family}$$

If the polynomials can be placed on the leaves of a tree so that when put average of descendants at nodes siblings have common interlacings
Interlacing Family of Polynomials

\[ \{ p_s \}_{s \in \{ \pm 1 \}^m} \text{ is an interlacing family} \]

If the polynomials can be placed on the leaves of a tree so that when put average of descendants at nodes siblings have common interlacings
Interlacing Family of Polynomials

Theorem:

There is an $s$ so that

$$\max\text{-root}(p_s(x)) \leq \max\text{-root}\left(\mathbb{E}_s[p_s(x)]\right)$$
An interlacing family

Theorem:
Let \( p_s(x) = \chi A_s(x) \)

\( \{p_s\}_{s \in \{\pm 1\}^m} \) is an interlacing family
Interlacing

\( p_1(x) \) and \( p_2(x) \) have a common interlacing iff

\[ \lambda p_1(x) + (1 - \lambda)p_2(x) \] is real rooted for all \( 0 \leq \lambda \leq 1 \)
To prove interlacing family

Let \( p_{s_1, \ldots, s_k}(x) = \mathbb{E}_{s_{k+1}, \ldots, s_m} \left[ p_{s_1, \ldots, s_m}(x) \right] \)
To prove interlacing family

Let \( p_{s_1, \ldots, s_k}(x) = \mathbb{E}_{s_{k+1}, \ldots, s_m} \left[ p_{s_1, \ldots, s_m}(x) \right] \)

Need to prove that for all \( s_1, \ldots, s_k, \lambda \in [0, 1] \)

\[
\lambda p_{s_1, \ldots, s_k, 1}(x) + (1 - \lambda)p_{s_1, \ldots, s_k, -1}(x)
\]

is real rooted
To prove interlacing family

Let \( p_{s_1, \ldots, s_k}(x) = \mathbb{E}_{s_{k+1}, \ldots, s_m} \left[ p_{s_1, \ldots, s_m}(x) \right] \)

Need to prove that for all \( s_1, \ldots, s_k, \lambda \in [0, 1] \)
\[
\lambda p_{s_1, \ldots, s_k, 1}(x) + (1 - \lambda)p_{s_1, \ldots, s_k, -1}(x)
\]
is real rooted

\( s_1, \ldots, s_k \) are fixed
\( s_{k+1} \) is 1 with probability \( \lambda \), -1 with probability \( 1 - \lambda \)
\( s_{k+2}, \ldots, s_m \) are uniformly \( \pm 1 \)
Generalization of Heilmann-Lieb

We prove

$$\mathbb{E}_{s \in \{\pm 1\}^m} \left[ p_s(x) \right]$$

is real rooted for every independent distribution on the entries of $s$. 
Generalization of Heilmann-Lieb

We prove

$$\mathbb{E}_{s \in \{\pm 1\}^m} \left[ p_s(x) \right]$$

is real rooted

for every independent distribution on the entries of $s$
Mixed Characteristic Polynomials

For $a_1, \ldots, a_n$ independently chosen random vectors

$$
\mathbb{E} \left[ \text{poly} \left( \sum_i a_i a_i^T \right) \right] = \mu(A_1, \ldots, A_n)
$$

is their mixed characteristic polynomial.

Theorem: Mixed characteristic polynomials are real rooted.

Mixed Characteristic Polynomials

For $a_1, \ldots, a_n$ independently chosen random vectors

$$E \left[ \text{poly} \left( \sum_i a_i a_i^T \right) \right] = \mu(A_1, \ldots, A_n)$$

is their mixed characteristic polynomial.

Obstacle: our matrix is a sum of random rank-2 matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$
Mixed Characteristic Polynomials

For $a_1, \ldots, a_n$ independently chosen random vectors

$$E \left[ \text{poly}(\sum_i a_i a_i^T) \right] = \mu(A_1, \ldots, A_n)$$

is their mixed characteristic polynomial.

Solution: add to the diagonal

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
Generalization of Heilmann-Lieb

We prove

$$\mathbb{E}_{s \in \{\pm 1\}^m} \left[ p_s(x) \right] \text{ is real rooted}$$

for every independent distribution on the entries of $s$

Implies $\chi_{A_s}(x)$ is an interlacing family
Generalization of Heilmann-Lieb

We prove

\[ \mathbb{E}_{s \in \{\pm 1\}^m} \left[ p_s(x) \right] \text{ is real rooted} \]

for every independent distribution on the entries of \( s \)

Implies \( \chi_{A_s}(x) \) is an interlacing family

Conclude there is an \( s \) so that

\[ \text{max-root} \left( \chi_{A_s}(x) \right) \leq 2\sqrt{d - 1} \]
Universal Covers

The universal cover of a graph $G$ is a tree $T$ of which $G$ is a quotient.
- vertices map to vertices
- edges map to edges
- homomorphism on neighborhoods

Is the tree of non-backtracking walks in $G$.

The universal cover of a $d$-regular graph is the infinite $d$-regular tree.
Quotients of Trees

$d$-regular Ramanujan as quotient of infinite $d$-ary tree

Spectral radius and norm of inf $d$-ary tree are

$$2\sqrt{d - 1}$$
Godsil’s Proof of Heilmann-Lieb

\[ T(G,\nu) : \text{the path tree of } G \text{ at } \nu \]
vertices are paths in \( G \) starting at \( \nu \)
edges to paths differing in one step
Godsil’s Proof of Heilmann-Lieb
Godsil’s Proof of Heilmann-Lieb

\( T(G,v) \): the path tree of \( G \) at \( v \)
vertices are paths in \( G \) starting at \( v \)
edges to paths differing in one step

Theorem:
The matching polynomial divides
the characteristic polynomial of \( T(G,v) \)
Godsil’s Proof of Heilmann-Lieb

\[ T(G,v) : \text{the path tree of } G \text{ at } v \]
vertices are paths in \( G \) starting at \( v \)
edges to paths differing in one step

**Theorem:**
The matching polynomial divides
the characteristic polynomial of \( T(G,v) \)

Is a subgraph of infinite tree,
so has smaller spectral radius
Quotients of Trees

(c,d)-biregular bipartite Ramanujan as quotient of infinite (c,d)-ary tree

Spectral radius \( \sqrt{d-1} + \sqrt{c-1} \)

For (c,d)-regular bipartite Ramanujan graphs

\( \sqrt{d-1} + \sqrt{c-1} \)
Questions

Non-bipartite Ramanujan Graphs of every degree?

Efficient constructions?

Explicit constructions?