Outline

Spectral Graph Theory: Understand graphs through eigenvectors and eigenvalues of associated matrices.

Electrical Graph Theory: Understand graphs through metaphor of resistor networks.

Heuristics
Algorithms
Theorems
Intuition
Spectral Graph Theory

Graph \( G = (V, E) \)

Matrix \( A \)
rows and cols indexed by \( V \)

Eigenvalues \( Av = \lambda v \)

Eigenvectors \( v : V \rightarrow \mathbb{R} \)
Spectral Graph Theory

Graph $G = (V, E)$

Matrix $A$
rows and cols indexed by $V$

Eigenvalues $Av = \lambda v$

Eigenvalues:

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

$A(i, j) = 1$ if $(i, j) \in E$

$-1 \ -0.618 \ 0.618 \ 1$
Example: Graph Drawing by the Laplacian
Example: Graph Drawing by the Laplacian
Example: Graph Drawing by the Laplacian

\[ L(i, j) = \begin{cases} 
-1 & \text{if } (i, j) \in E \\
\deg(i) & \text{if } i = j \\
0 & \text{otherwise}
\end{cases} \]
Example: Graph Drawing by the Laplacian

\[ L(i, j) = \begin{cases} 
-1 & \text{if } (i, j) \in E \\
\deg(i) & \text{if } i = j \\
0 & \text{otherwise}
\end{cases} \]

Eigenvalues 0, 1.53, 1.53, 3, 3.76, 3.76, 5, 5.7, 5.7

Let \( x, y \in \mathbb{R}^V \) span eigenspace of eigenvalue 1.53
Example: Graph Drawing by the Laplacian

Plot vertex $i$ at $(x(i), y(i))$

Draw edges as straight lines
Laplacian: natural quadratic form on graphs

\[ x^T L x = \sum_{(i,j) \in E} (x(i) - x(j))^2 \]

\[ L = D - A \]

where D is diagonal matrix of degrees

\[
\begin{pmatrix}
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{pmatrix}
\]
Laplacian: fast facts

\[ x^T L x = \sum_{(i,j) \in E} (x(i) - x(j))^2 \]

\[ L1 = 0 \quad \text{zero is an eigenvalue} \]

\[ 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \]

Connected if and only if \( \lambda_2 > 0 \)

Fiedler (‘73) called \( \lambda_2 \)
\[ \text{“algebraic connectivity of a graph”} \]
The further from 0, the more connected.
Drawing a graph in the line (Hall ’70)

map $V \rightarrow \mathbb{R}$

minimize $\sum_{(i,j) \in E} (x(i) - x(j))^2 = x^T L x$

trivial solution: $x = 1$  So, require $x \perp 1$, $\|x\| = 1$

Solution $x = v_2$

Atkins, Boman, Hendrickson ’97:
Gives correct drawing for graphs like
Courant-Fischer definition of eigvals/vecs

$$\lambda_1 = \min_{x \neq 0} \frac{x^T L x}{x^T x}$$

$$v_1 = \arg \min_{x \neq 0} \frac{x^T L x}{x^T x}$$
Courant-Fischer definition of eigvals/vecs

\[ \lambda_1 = \min_{x \neq 0} \frac{x^T Lx}{x^T x} \quad \text{and} \quad v_1 = \arg \min_{x \neq 0} \frac{x^T Lx}{x^T x} \]

\[ \lambda_2 = \min_{x \perp v_1} \frac{x^T Lx}{x^T x} \quad \text{and} \quad v_2 = \arg \min_{x \perp v_1} \frac{x^T Lx}{x^T x} \]

\( \text{(here } v_1 = 1 ) \)
Courant-Fischer definition of eigvals/vecs

\[ \lambda_1 = \min_{x \neq 0} \frac{x^T Lx}{x^T x} \quad \text{and} \quad v_1 = \arg \min_{x \neq 0} \frac{x^T Lx}{x^T x} \]

\[ \lambda_2 = \min_{x \perp v_1} \frac{x^T Lx}{x^T x} \quad \text{and} \quad v_2 = \arg \min_{x \perp v_1} \frac{x^T Lx}{x^T x} \]

(Here \( v_1 = 1 \))

\[ \lambda_k = \min_{S \text{ of dim } k} \max_{x \in S} \frac{x^T Lx}{x^T x} \]

\[ v_k = \arg \min_{x \perp v_1, \ldots, v_{k-1}} \frac{x^T Lx}{x^T x} \]
Drawing a graph in the plane (Hall ’70)

\[ \text{map} \quad V \rightarrow \mathbb{R}^2 \quad \vec{x}(i) \in \mathbb{R}^2 \]

minimize \[ \sum_{(i, j) \in E} (\text{dist}(\vec{x}(i), \vec{x}(j)))^2 \]
Drawing a graph in the plane (Hall ’70)

map \ V \rightarrow \mathbb{R}^2 \quad \vec{x}(i) \in \mathbb{R}^2

minimize \quad \sum_{(i,j) \in E} (\text{dist}(\vec{x}(i), \vec{x}(j)))^2

trivial solution: \quad \vec{x}(i) = (1, 1)

So, require \quad \vec{x}_1, \vec{x}_2 \perp 1
Drawing a graph in the plane (Hall ’70)

map \ V \rightarrow \mathbb{R}^2 \quad \vec{x}(i) \in \mathbb{R}^2

minimize \quad \sum_{(i,j) \in E} (\text{dist}(\vec{x}(i), \vec{x}(j)))^2

trivial solution: \quad \vec{x}(i) = (1, 1)

So, require \quad \vec{x}_1, \vec{x}_2 \perp 1

diagonal solution: \quad \vec{x}(i) = (v_2(i), v_2(i))

So, require \quad \vec{x}_1 \perp \vec{x}_2

Solution \quad \vec{x}(i) = (v_2(i), v_3(i)) \quad \text{up to rotation}
A Graph
Drawing of the graph using $v_2, v_3$

Plot vertex $i$ at $(v_2(i), v_3(i))$
The Airfoil Graph, original coordinates
The Airfoil Graph, spectral coordinates
The Airfoil Graph, spectral coordinates
Spectral drawing of Streets in Rome
Spectral drawing of Erdos graph:
edge between co-authors of papers
Dodecahedron

Best embedded by first three eigenvectors
Intuition: Graphs as Spring Networks

edges -> ideal linear springs
weights -> spring constants (k)

Physics: when stretched to length x, force is kx
potential energy is kx^2/2

Nail down some vertices, let rest settle
Intuition: Graphs as Spring Networks

Nail down some vertices, let rest settle

\[ x(i) \]

Physics: minimizes total potential energy

\[
\sum_{(i,j) \in E} (x(i) - x(j))^2 = x^T L x
\]

subject to boundary constraints (nails)
Intuition: Graphs as Spring Networks

Nail down some vertices, let rest settle

\[ x(i) \]

Physics: energy minimized when non-fixed vertices are averages of neighbors

\[ \vec{x}(i) = \frac{1}{d_i} \sum_{(i,j) \in E} \vec{x}(j) \]
Tutte’s Theorem ‘63

If nail down a face of a planar 3-connected graph, get a planar embedding!
Spectral graph drawing: Tutte justification

Condition for eigenvector \( Lx = \lambda x \)

Gives \( x(i) = \frac{1}{d_i - \lambda} \sum_{(i,j) \in E} x(j) \) for all \( i \)

\( \lambda \) small says \( x(i) \) near average of neighbors
Spectral graph drawing: Tutte justification

Condition for eigenvector $Lx = \lambda x$

Gives $x(i) = \frac{1}{d_i - \lambda} \sum_{(i,j) \in E} x(j)$ for all $i$

$\lambda$ small says $x(i)$ near average of neighbors

For planar graphs:

$\lambda_2 \leq 8d/n$ \[S-Teng \ '96\]

$\lambda_3 \leq O(d/n)$ \[Kelner-Lee-Price-Teng \ '09\]
Small eigenvalues are not enough

Plot vertex \( i \) at \((v_3(i), v_4(i))\)
Graph Partitioning
Spectral Graph Partitioning
[Donath-Hoffman ‘72, Barnes ‘82, Hagen-Kahng ‘92]

\[ S = \{ i : v_2(i) \leq t \} \text{ for some } t \]
Measuring Partition Quality: Conductance

\[ \Phi(S) = \frac{\text{\# edges leaving } S}{\text{sum of degrees in } S} \]

For \( \deg(S) \leq \frac{\deg(V)}{2} \)
Spectral Image Segmentation (Shi-Malik '00)
Spectral Image Segmentation (Shi-Malik ‘00)
Spectral Image Segmentation (Shi-Malik ‘00)
Spectral Image Segmentation (Shi-Malik ‘00)
Spectral Image Segmentation (Shi-Malik ‘00)

edge weight $e^{-\text{diff}(\text{pixel}_i, \text{pixel}_j)^2/t^2}$
The second eigenvector
Second eigenvector cut
Third Eigenvector
Fourth Eigenvector
Cheeger’s Inequality  [Cheeger ‘70]
[Alon-Milman ‘85, Jerrum-Sinclair ‘89, Diaconis-Stroock ‘91]

For Normalized Laplacian: \( \mathcal{L} = D^{-1/2}LD^{-1/2} \)

\[ \frac{\lambda_2}{2} \leq \min_S \Phi(S) \leq \sqrt{2\lambda_2} \]

And, is a spectral cut for which

\[ \Phi(S) \leq \sqrt{2\lambda_2} \]
McSherry’s Analysis of Spectral Partitioning

Divide vertices into $S$ and $T$
Place edges at random with

\[
\begin{align*}
\Pr [S-S \text{ edge}] &= p \\
\Pr [T-T \text{ edge}] &= p \\
\Pr [S-T \text{ edge}] &= q
\end{align*}
\]

$q < p$
McSherry’s Analysis of Spectral Partitioning

\[ E[ A ] = \begin{pmatrix} p & q \\ q & p \end{pmatrix} \begin{cases} \} S \\ \} T \end{cases} \]
McSherry’s Analysis of Spectral Partitioning

\[ \mathbf{E}[A] = \begin{pmatrix} p & q \\ q & p \end{pmatrix} \begin{cases} \{S\} \\ \{T\} \end{cases} \]

\(v_2(\mathbf{E}[L])\) is positive const on S, negative const on T

View \(A\) as perturbation of \(\mathbf{E}[A]\)
and \(L\) as perturbation of \(\mathbf{E}[L]\)
**McSherry’s Analysis of Spectral Partitioning**

\[ v_2(\mathbf{E}[L]) \text{ is negative const on } S, \text{ positive const on } T \]

View \( A \) as perturbation of \( \mathbf{E}[A] \)
and \( L \) as perturbation of \( \mathbf{E}[L] \)

Random Matrix Theory [Füredi-Komlós ‘81, Vu ‘07]

With high probability \( \| L - \mathbf{E}[L] \| \) small

Perturbation Theory for Eigenvectors implies

\[ v_2(L) \approx v_2(\mathbf{E}[L]) \]
Spectral graph coloring from high eigenvectors

Embedding of dodecahedron by 19\textsuperscript{th} and 20\textsuperscript{th} eigvecs.
Spectral graph coloring from high eigenvectors

Coloring 3-colorable random graphs [Alon-Kahale ’97]
Independent Sets

S is independent if there are no edges between vertices in S
Independent Sets

S is independent if there are no edges between vertices in S

Hoffman’s Bound: if every vertex has degree $d$

$$|S| \leq n \left(1 - \frac{d}{\lambda_n}\right)$$
Networks of Resistors

Ohm’s laws gives $i = v / r$

In general, $i = L_G v$ with $w_{(u,v)} = 1/r_{(u,v)}$

Minimize dissipated energy $v^T L_G v$
Networks of Resistors

Ohm’s laws gives $i = \frac{v}{r}$

In general, $i = L_G v$ with $w_{a,b} = \frac{1}{r_{a,b}}$

Minimize dissipated energy $v^T L_G v$

By solving Laplacian

$0V$

$0.375V$

$0.5V$

$0.5V$

$1V$

$0.625V$
Electrical Graph Theory

Considers flows in graphs

Allows comparisons of graphs, and embedding of one graph within another.

Relative Spectral Graph Theory
Effective Resistance

Resistance of entire network, measured between a and b.

Ohm’s law: \( r = \frac{v}{i} \)

\( R_{\text{eff}}(a, b) = \frac{1}{\text{(current flow at one volt)}} \)
Effective Resistance

Resistance of entire network, measured between \( a \) and \( b \).

Ohm's law: \( r = \frac{v}{i} \)

\[ R_{\text{eff}}(a, b) = \frac{1}{(\text{current flow at one volt})} = \text{voltage difference to flow 1 unit} \]
Effective Resistance

$$R_{\text{eff}}(a,b) = \text{voltage difference to flow 1 unit}$$

Vector of one unit flow has 1 at $a$, 
-1 at $b$, 
0 elsewhere

$$i_{a,b} = e_a - e_b$$

Voltages required by this flow are given by

$$v_{a,b} = L_G^{-1} i_{a,b}$$
Effective Resistance

\[ R_{\text{eff}}(a, b) = \text{voltage difference of unit flow} \]

Voltages required by unit flow are given by

\[ v_{a,b} = L_G^{-1} i_{a,b} \]

Voltage difference is

\[
\begin{align*}
  v_{a,b}(a) - v_{a,b}(b) &= (e_a - e_b)^T v_{a,b} \\
  &= (e_a - e_b)^T L_G^{+}(e_a - e_b)
\end{align*}
\]
Effective Resistance Distance

Effective resistance is a distance lower when there are more short paths.

Equivalent to commute time distance: expected time for a random walk from \(a\) to reach \(b\) and then return to \(a\).

See Doyle and Snell, *Random Walks and Electrical Networks*.
Relative Spectral Graph Theory

For two connected graphs G and H with the same vertex set, consider

$$L_G L_H^{-1}$$

work orthogonal to nullspace
or use pseudoinverse

Allows one to compare G and H
Relative Spectral Graph Theory

For two connected graphs $G$ and $H$, consider

$$L_G L_H^{-1} = I_{n-1}$$

if and only if $G = H$
Relative Spectral Graph Theory

For two connected graphs $G$ and $H$, consider

$$L_G L_H^{-1} \approx I_{n-1}$$

if and only if $G \approx H$
Relative Spectral Graph Theory

For two connected graphs $G$ and $H$, consider

\[ \frac{1}{1 + \epsilon} \leq \text{eigs}(L_G L_H^{-1}) \leq 1 + \epsilon \]

if and only if for all $x \in \mathbb{R}^V$

\[ \frac{1}{1 + \epsilon} \leq \frac{x^T L_G x}{x^T L_H x} \leq 1 + \epsilon \]
Relative Spectral Graph Theory

\[
\frac{1}{1 + \epsilon} \leq \frac{x^T L_G x}{x^T L_H x} \leq 1 + \epsilon
\]

In particular, for \( x(a) = \begin{cases} 1 & a \in S \\ 0 & a \not\in S \end{cases} \)

\[
x^T L_G x = \sum_{(a,b) \in E} (x(a) - x(b))^2 = |E(S, V - S)|
\]
Relative Spectral Graph Theory

\[
\frac{1}{1 + \epsilon} \leq \frac{x^T L_G x}{x^T L_H x} \leq 1 + \epsilon
\]

For all \( S \subset V \)

\[
\frac{1}{1 + \epsilon} \leq \frac{|E_G(S, V - S)|}{|E_H(S, V - S)|} \leq 1 + \epsilon
\]
Expanders Approximate Complete Graphs

Expanders:

\(d\)-regular graphs on \(n\) vertices

high conductance

random walks mix quickly

weak expanders: eigenvalues bounded from 0

strong expanders: all eigenvalues near \(d\)
Expanders Approximate Complete Graphs

For $G$ the complete graph on $n$ vertices.
all non-zero eigenvalues of $L_G$ are $n$.

For $x \perp 1$, $\|x\| = 1$ \quad $x^T L_G x = n$
Expanders Approximate Complete Graphs

For $G$ the complete graph on $n$ vertices.
all non-zero eigenvalues of $L_G$ are $n$.

For $x \perp 1, \|x\| = 1 \quad x^T L_G x = n$

For $H$ a $d$-regular strong expander,
all non-zero eigenvalues of $L_H$ are close to $d$.

For $x \perp 1, \|x\| = 1 \quad x^T L_H x \in [\lambda_2, \lambda_n] \approx d$
Expanders Approximate Complete Graphs

For \( G \) the complete graph on \( n \) vertices, all non-zero eigenvalues of \( L_G \) are \( n \).

For \( x \perp \mathbf{1}, \|x\| = 1 \quad x^T L_G x = n \)

For \( H \) a \( d \)-regular strong expander, all non-zero eigenvalues of \( L_H \) are close to \( d \).

For \( x \perp \mathbf{1}, \|x\| = 1 \quad x^T L_H x \approx d \)

\( \frac{n}{d} H \) is a good approximation of \( G \)
Sparse approximations of every graph

\[
\frac{1}{1 + \epsilon} \leq \frac{x^T L_G x}{x^T L_H x} \leq 1 + \epsilon
\]

For every G,

there is an H with \((2 + \epsilon)^2 n / \epsilon^2\) edges

[Batson-S-Srivastava]

Can find an H with \(O(n \log n / \epsilon^2)\) edges

in nearly-linear time.

[S-Srivastava]
Sparsification by Random Sampling [S-Srivastava]

Include edge \((u, v)\) with probability

\[ p_{u,v} \sim w_{u,v} R_{\text{eff}}(u, v) \]

If include edge, give weight \( w_{u,v} / p_{u,v} \)

Analyze by Rudelson’s
  concentration of random sums of rank-1 matrices
Approximating a graph by a tree

Alon, Karp, Peleg, West ‘91: measure the stretch
Approximating a graph by a tree

Alon, Karp, Peleg, West ‘91: measure the stretch
Approximating a graph by a tree

Alon, Karp, Peleg, West ‘91: measure the stretch

\[ \text{stretch}_T(i, j) = \text{dist}_T(i, j) \]
Approximating a graph by a tree

Alon, Karp, Peleg, West ‘91: measure the stretch

\[
\text{stretch}_T(G) = \sum_{(i,j) \in G} \text{dist}_T(i, j)
\]
Low-Stretch Spanning Trees

For every $G$ there is a $T$ with

$$\text{stretch}_T(G) \leq m^{1+o(1)}$$

where $m = |E|$

(Alon-Karp-Peleg-West ’91)

$$\text{stretch}_T(G) \leq O(m \log m \log^2 \log m)$$

(Elkin-Emek-S-Teng ’04, Abraham-Bartal-Neiman ’08)

Conjecture: $\text{stretch}_T(G) \leq m \log_2 m$
Algebraic characterization of stretch [S-Woo ’09]

\[ \text{stretch}_T(G) = \text{Trace}[L_G L_T^{-1}] \]
Algebraic characterization of stretch [S-Woo ’09]

\[ \text{stretch}_T(G) = \text{Trace}[L_G L_T^{-1}] \]

Resistances in series sum

In trees, resistance is distance.

\[ U : \begin{array}{cccc}
0 & 1 & 1 & 1 \\
\text{a} & 1 & \text{T} & \text{b} \\
2 & 3 & 4 & 4 \\
\end{array} \]
Algebraic characterization of stretch [S-Woo ’09]

\[ \text{stretch}_T(G) = \text{Trace}[L_G L_T^{-1}] \]

\[ x^T L_G x = \sum_{(a,b) \in E} (x(a) - x(b))^2 \]

\[ = \sum_{(a,b) \in E} ((e_a - e_b)^T x)^2 \]

\[ = \sum_{(a,b) \in E} x^T (e_a - e_b)(e_a - e_b)^T x \]

\[ = x^T \left( \sum_{(a,b) \in E} (e_a - e_b)(e_a - e_b)^T \right) x \]
Algebraic characterization of stretch [S-Woo ’09]

\[ \text{stretch}_T(G) = \text{Trace}[L_G L_T^{-1}] \]

\[
\text{Trace}[L_G L_T^{-1}] = \sum_{(a,b) \in E} \text{Trace}[(e_a - e_b)(e_a - e_b)^T L_T^{-1}]
\]

\[ = \sum_{(a,b) \in E} \text{Trace}[(e_a - e_b)^T L_T^{-1}(e_a - e_b)] \]

\[ = \sum_{(a,b) \in E} (e_a - e_b)^T L_T^{-1}(e_a - e_b) \]
Algebraic characterization of stretch [S-Woo ’09]

\[ \text{stretch}_T(G) = \text{Trace}[L_G L_T^{-1}] \]

\[ \sum_{(a,b) \in E} (e_a - e_b)^T L_T^{-1} (e_a - e_b) = \sum_{(a,b) \in E} \text{R}_{\text{eff}}(a,b) \]

\[ = \sum_{(a,b) \in E} \text{stretch}_T(a,b) \]
Notable Things I’ve left out

Behavior under graph transformations
Graph Isomorphism
Random Walks and Diffusion
PageRank and Hits
Matrix-Tree Theorem
Special Graphs
   (Cayley, Strongly-Regular, etc.)
Diameter bounds
Colin de Verdière invariant
Discretizations of Manifolds
The next two talks

Tomorrow:
Solving equations in Laplacians in nearly-linear time.

Preconditioning
Sparsification
Low-Stretch Spanning Trees
Local graph partitioning
The next two talks

Thursday:
Existence of sparse approximations.

A theorem in linear algebra and some of its connections.